

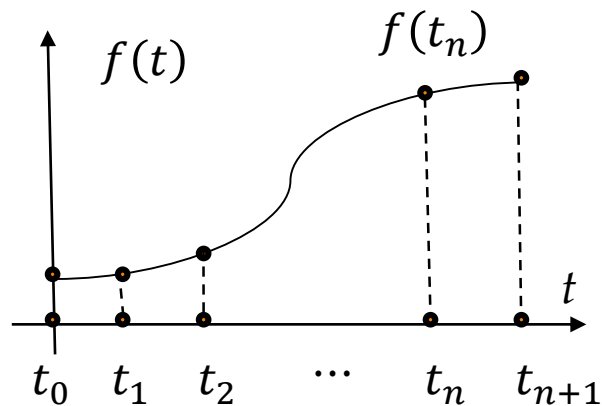
Modele systemów dynamicznych



Wykład 7. Dyskretyzacja sygnałów ciągłych. Opis obiektu ciągłego sterowanego sygnałem dyskretnym

Dyskretyzacja sygnału ciągłego

$f(t), f(t) = 0$ dla $t < 0$



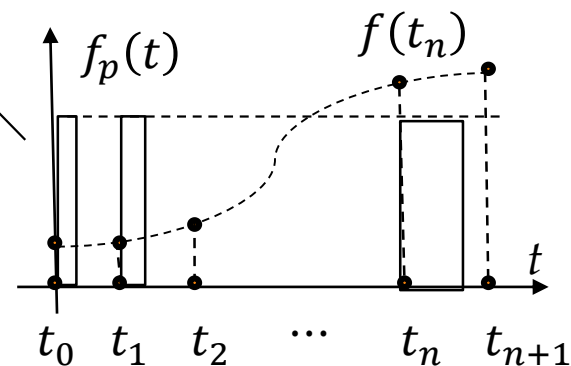
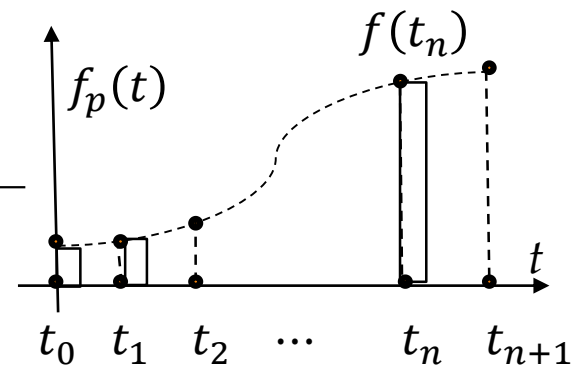
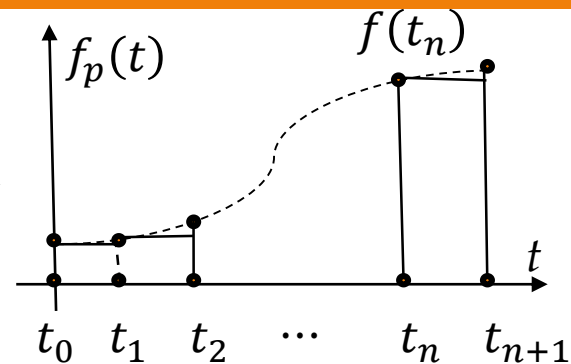
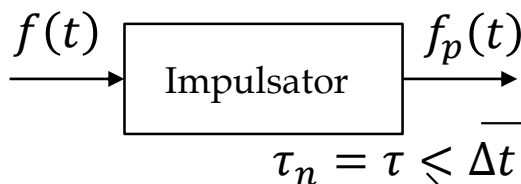
$$t_n = t_0 + n\Delta t$$

$$t_0 = 0, n = 0, 1, 2, \dots$$

Δt – okres dyskretyzacji

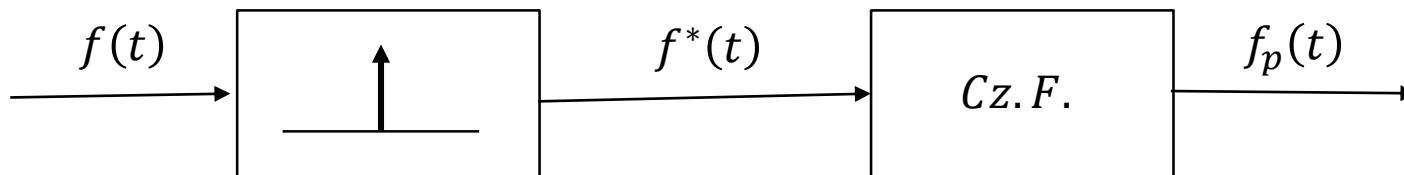
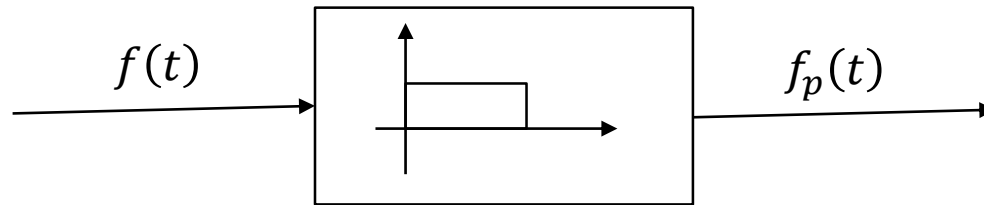
τ_n – czas trwania impulsu

$\tau_n = \tau = \Delta t$ – stałe
Ekstrapolator – E_0



$\tau_n \sim f(t_n), \tau_n$ – zmienne

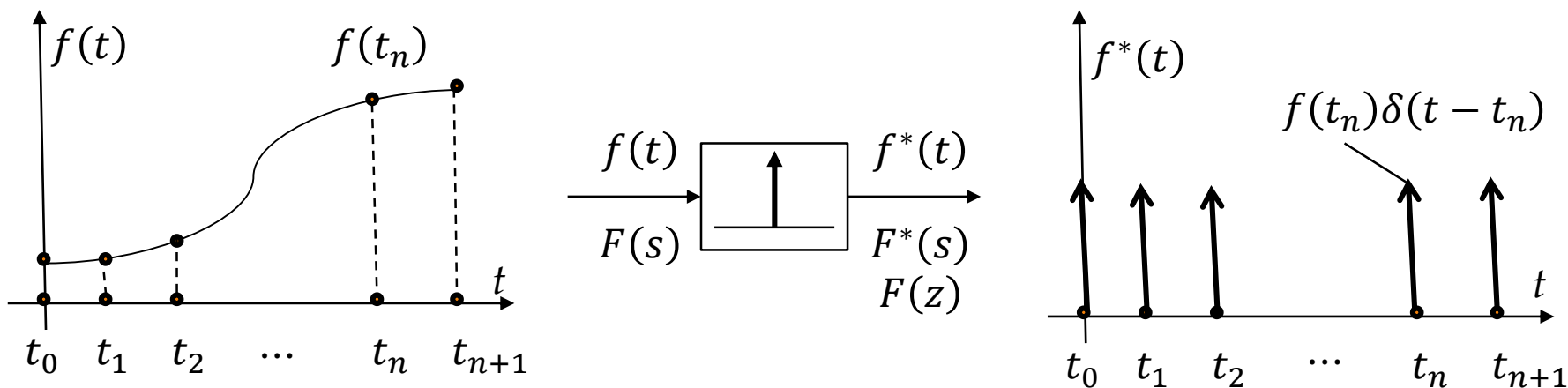
Impulsator



Impulsator
idealny

Człon
Formułujący

Impulsator idealny



$f^*(t) \sim$ ciąg impulsów $f(t_0)\delta(t - t_0), f(t_1)\delta(t - t_1), \dots, f(t_n)\delta(t - t_n)$

Oznaczmy przez: $S(t) = \sum_{n=0}^{\infty} \delta(t - t_n)$

Wówczas:

$$f^*(t) = f(t)S(t) = f(t) \sum_{n=0}^{\infty} \delta(t - t_n) = \sum_{n=0}^{\infty} f(t) \delta(t - t_n)$$

Relacja pomiędzy $F(s)$ i $F(z)$ *impulsator idealny*

$f(t)$ zamieniono na ciąg impulsów $f(t_0)\delta(t - t_0), f(t_1)\delta(t - t_1), \dots, f(t_n)\delta(t - t_n)$

$$f(t) \text{ po impulsatorze idealnym } f^*(t) = f(t)S(t) = f(t) \sum_{n=0}^{\infty} \delta(t - t_n) = \sum_{n=0}^{\infty} f(t)\delta(t - t_n)$$

$$F^*(s) = \mathcal{L}[f^*(t)] = \int_0^{\infty} \sum_{n=0}^{\infty} f(t)\delta(t - t_n)e^{-st} dt = \sum_{n=0}^{\infty} \int_0^{\infty} f(t)\delta(t - t_n)e^{-st} dt = \sum_{n=0}^{\infty} f(t_n)e^{-st_n}$$

ponieważ: $t_n = t_0 + n\Delta t$, $t_0 = 0$ zatem: $t_n = n\Delta t$ stąd :

$$F^*(s) = \sum_{n=0}^{\infty} f(t_n)e^{-st_n} = \sum_{n=0}^{\infty} f(n\Delta t)e^{-sn\Delta t}$$

oznaczymy przez: $f(n\Delta t) \stackrel{df}{=} f_n$, $e^{s\Delta t} \stackrel{df}{=} z$ stąd

$$F(s) \xrightarrow{\text{po dyskretyzacji}} F^*(s) = \sum_{n=0}^{\infty} f(n\Delta t)e^{-sn\Delta t} = \sum_{n=0}^{\infty} f_n z^{-n} = F(z)$$

Relacja pomiędzy $F(s)$ i $F(z)$ *impulsator idealny*

$$F(s) \rightarrow f(t) = \mathcal{L}'[F(s)] \rightarrow f(n\Delta t) = f_n \rightarrow F(z) = \mathcal{Z}[f_n]$$

$$f(t) = \delta(t) = \begin{cases} \infty & \text{dla } t = 0 \\ 0 & \text{dla } t \neq 0 \end{cases} \rightarrow f_n = \delta(n\Delta t) = \delta_n = \begin{cases} 1 & \text{dla } n = 0 \\ 0 & \text{dla } n \neq 0 \end{cases}$$

$$F(z) = \mathcal{Z}[\delta_n] = \sum_{n=0}^{\infty} \delta_n z^{-n} = 1$$

$$f(t) = 1(t) = \begin{cases} 1 & \text{dla } t \geq 0 \\ 0 & \text{dla } t < 0 \end{cases} \rightarrow f_n = 1(n\Delta t) = 1_n = \begin{cases} 1 & \text{dla } n \geq 0 \\ 0 & \text{dla } n < 0 \end{cases}$$

$$F(z) = \mathcal{Z}[1_n] = \sum_{n=0}^{\infty} 1_n z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

Relacja pomiędzy $F(s)$ i $F(z)$

impulsator idealny

$$f(t) = 1(t)e^{at} \rightarrow f_n = f(n\Delta t) = 1_n e^{an\Delta t} = 1_n (e^{a\Delta t})^n$$

$$F(z) = \mathcal{Z} \left[1_n (e^{a\Delta t})^n \right] = \sum_{n=0}^{\infty} (e^{a\Delta t})^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{e^{a\Delta t}}{z} \right)^n = \frac{1}{1 - \frac{e^{a\Delta t}}{z}} = \frac{z}{z - e^{a\Delta t}}$$

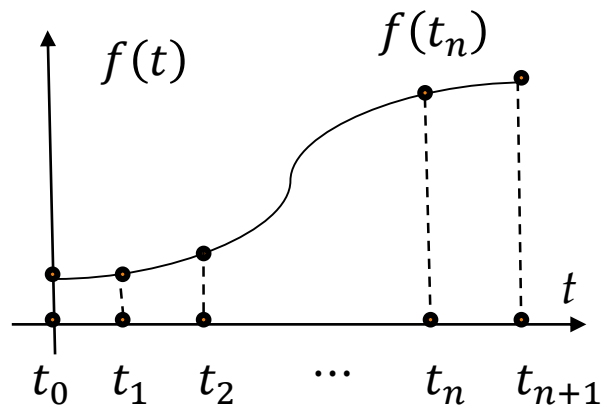
$$f(t) = 1(t)t \rightarrow f_n = 1_n n\Delta t \rightarrow F(z) = \frac{\Delta t z}{(z-1)^2}$$

$$f(t) = 1(t)\sin \beta t \rightarrow f_n = 1_n \sin \beta n\Delta t \rightarrow F(z) = \frac{z \sin \beta \Delta t}{z^2 - 2 \cos \beta \Delta t + 1}$$

$$f(t) = 1(t)\cos \beta t \rightarrow f_n = 1_n \cos \beta n\Delta t \rightarrow F(z) = \frac{z^2 - 2z \cos \beta \Delta t}{z^2 - 2 \cos \beta \Delta t + 1}$$

Dyskretyzacja sygnału ciągłego

$f(t), f(t) = 0$ dla $t < 0$

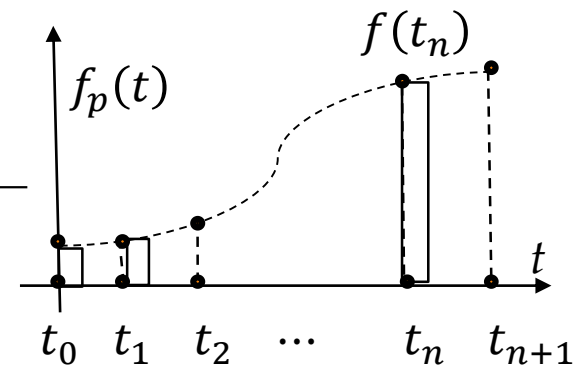
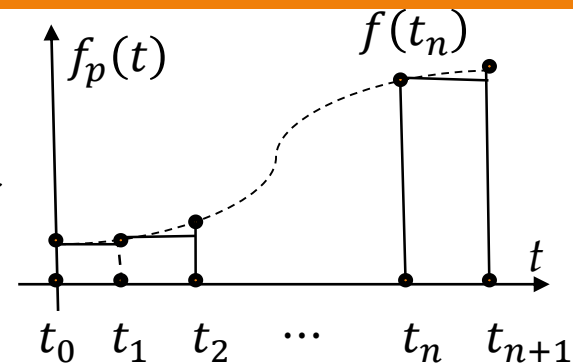
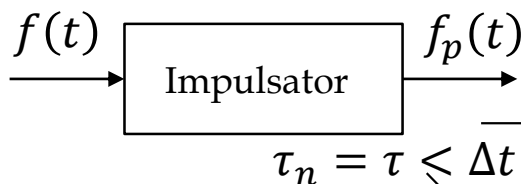


$$t_n = t_0 + n\Delta t$$

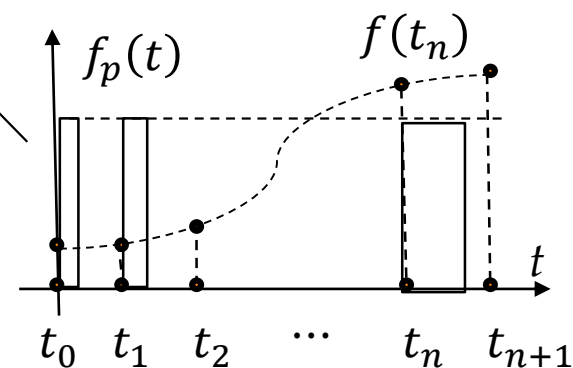
$$t_0 = 0$$

τ_n – czas trwania impulsu

$\tau_n = \tau = \Delta t$ – stałe
Ekstrapolator – E_0

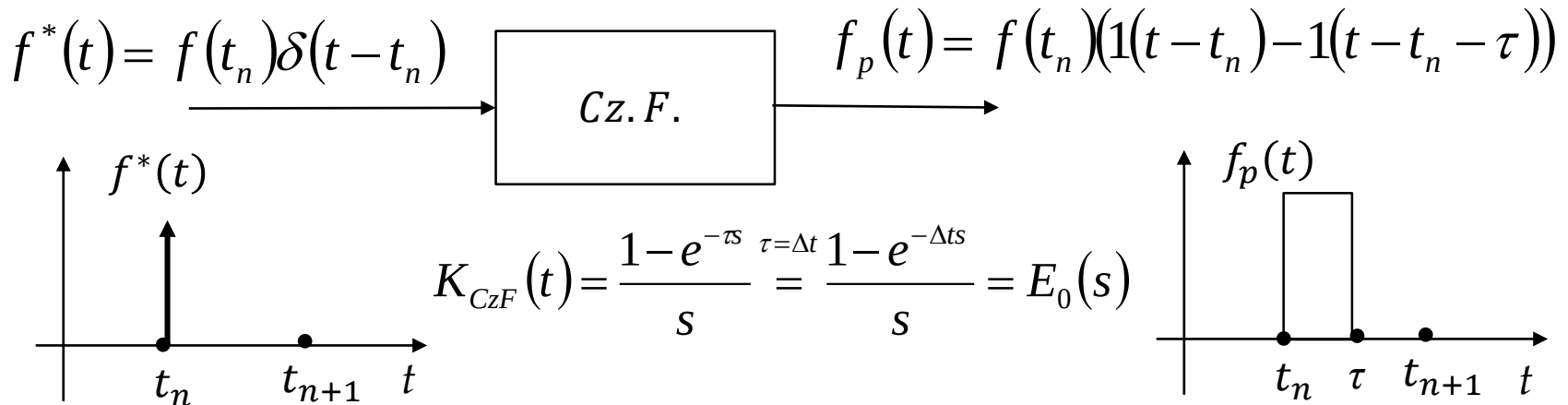


$\tau_n \sim f(t_n), \tau_n$ – zmienne



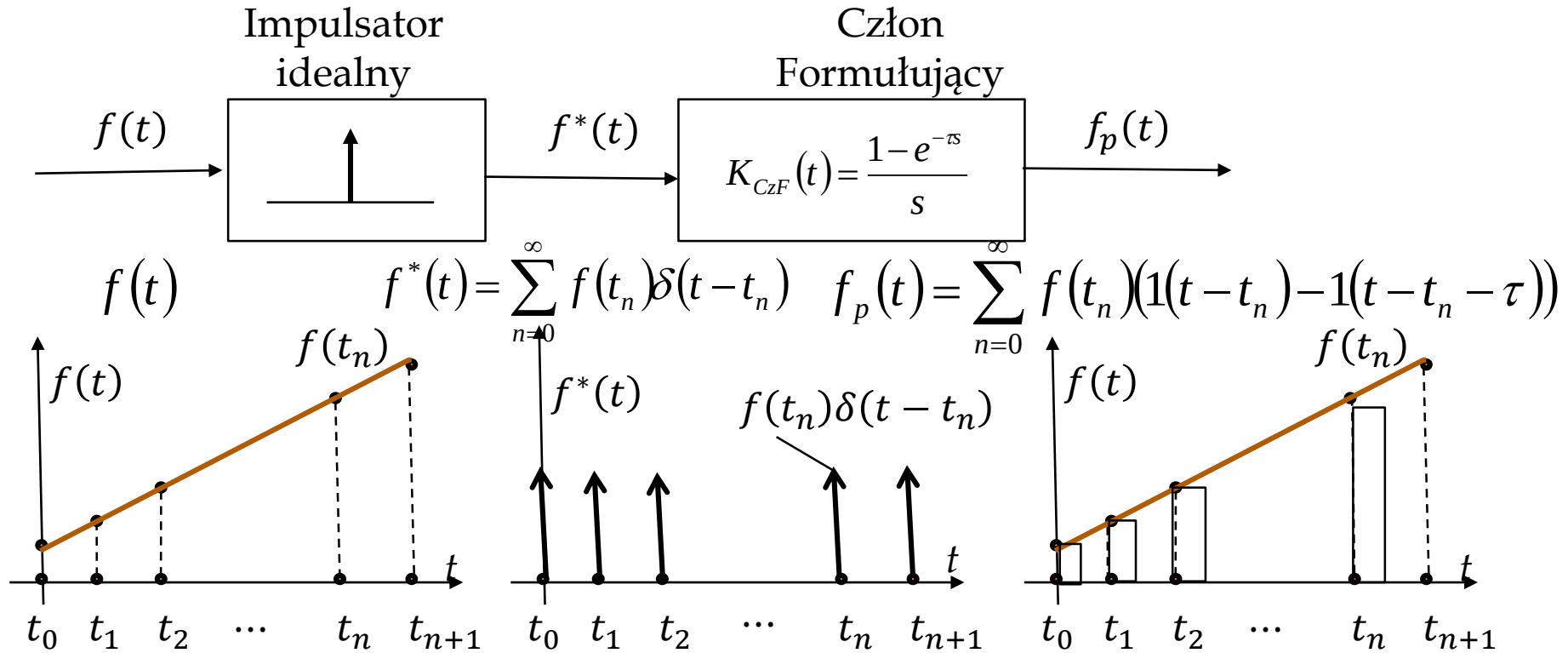
Człon formuujący

szerokość impulsu – $\tau < \Delta t$ – stałe



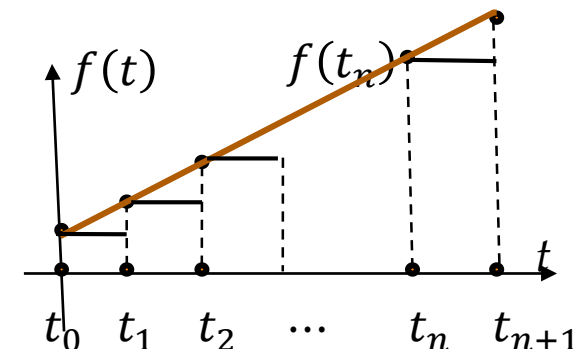
$$\begin{aligned}
 K_{CzF}(s) &= \frac{\mathcal{L}[f_p(t)]}{\mathcal{L}[f^*(t)]} = \frac{\mathcal{L}[f(t_n)(1(t - t_n) - 1(t - t_n - \tau))]}{\mathcal{L}[f(t_n)\delta(t - t_n)]} = \\
 &= \frac{f(t_n) \left(\frac{1}{s} e^{-t_n s} - \frac{1}{s} e^{-(t_n + \tau)s} \right)}{f(t_n) e^{-t_n s}} = \frac{1}{s} - \frac{1}{s} e^{-\tau s} = \frac{1 - e^{-\tau s}}{s}
 \end{aligned}$$

Impulsator – $\tau < \Delta t$ – stałe



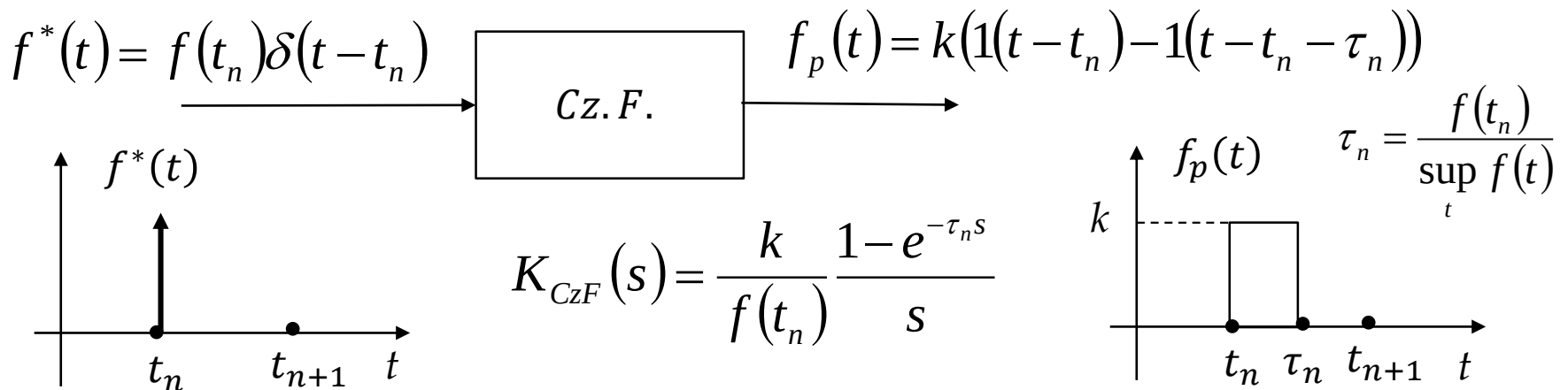
Szczególny przypadek $\tau = \Delta t$ - Eksplorator rzędu zerowego

$$K_{CzF}(t) = \frac{1 - e^{-\tau s}}{s} \stackrel{\tau = \Delta t}{=} \frac{1 - e^{-\Delta t s}}{s} = E_0(s)$$



Człon formułujący

szerokość impulsu – $\tau_n < \Delta t$ – zmienne

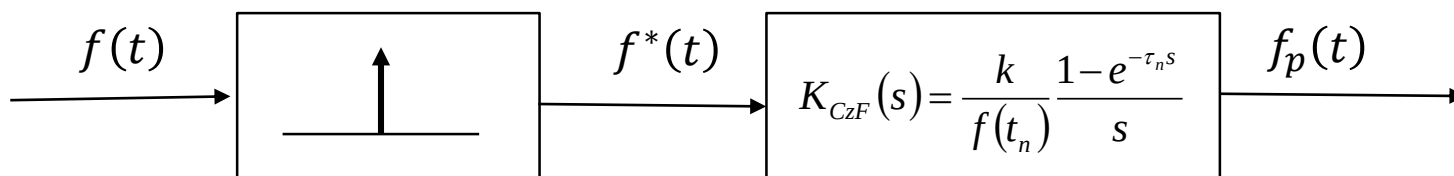


$$\begin{aligned}
 K_{CzF}(s) &= \frac{\mathcal{L}[f_p(t)]}{\mathcal{L}[f^*(t)]} = \frac{\mathcal{L}[k(1(t - t_n) - 1(t - t_n - \tau_n))]}{\mathcal{L}[f(t_n)\delta(t - t_n)]} = \\
 &= \frac{k\left(\frac{1}{s}e^{-t_n s} - \frac{1}{s}e^{-(t_n + \tau_n)s}\right)}{f(t_n)e^{-t_n s}} = \frac{k}{f(t_n)}\left(\frac{1}{s} - \frac{1}{s}e^{-\tau_n s}\right) = \frac{k}{f(t_n)} \frac{1 - e^{-\tau_n s}}{s}
 \end{aligned}$$

Impulsator – $\tau_n < \Delta t$ – zmienne

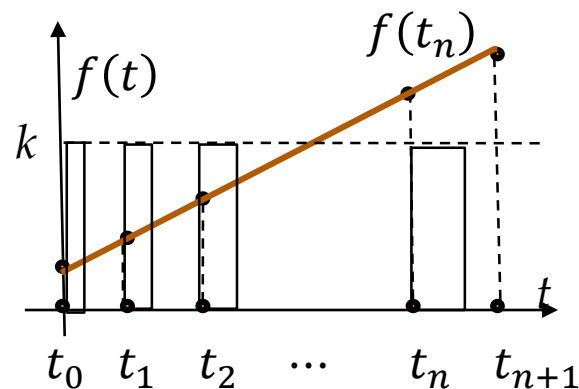
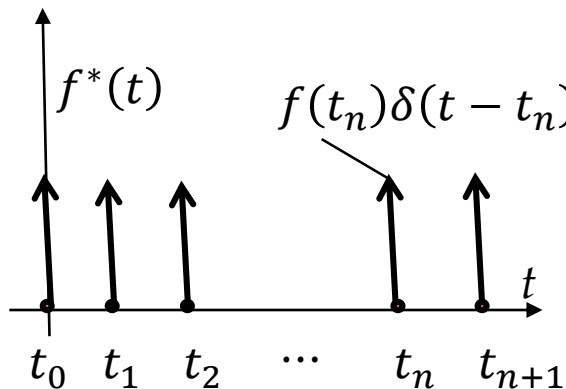
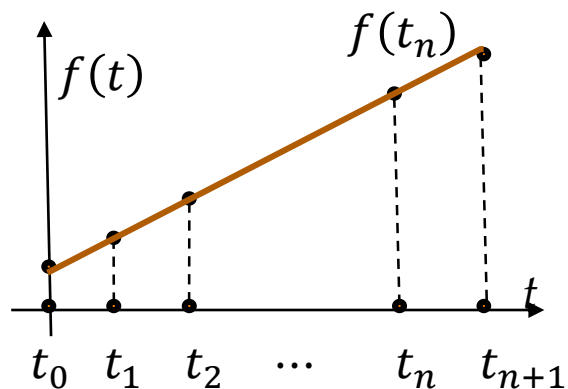
Impulsator
idealny

Człon
Formułujący



$$f(t)$$

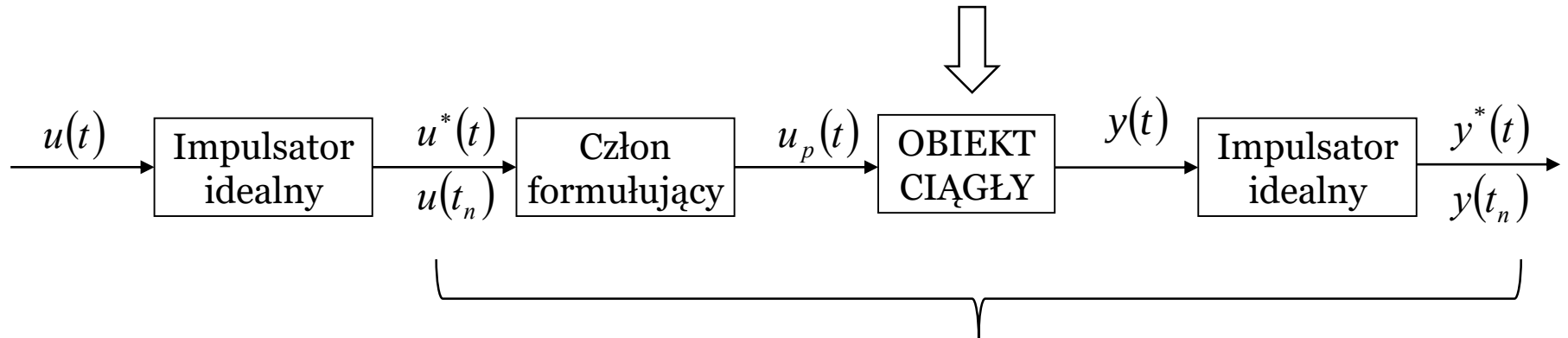
$$f^*(t) = \sum_{n=0}^{\infty} f(t_n) \delta(t - t_n) \quad f_p(t) = \sum_{n=0}^{\infty} f(t_n) (1(t - t_n) - 1(t - t_n - \tau_n))$$



Relacje pomiędzy opisami

Równania stanu

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$



$$x_{n+1} = \bar{A}x_n + \bar{B}u_n$$

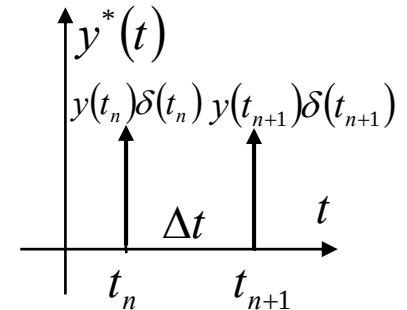
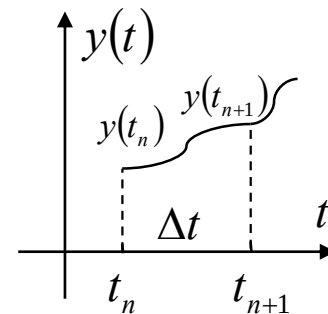
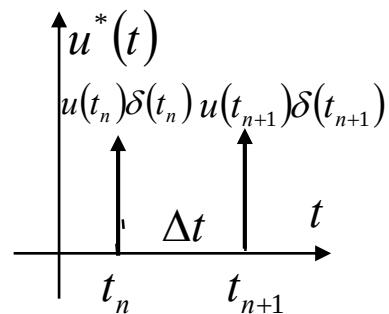
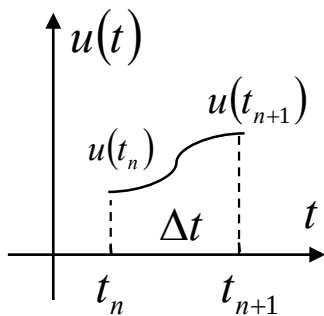
$$y_n = \bar{C}x_n + \bar{D}u_n$$

Relacje pomiędzy opisami

Równania stanu

Opis przy pomocy równań stanu – impulsator idealny bez członu formułującego

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad x_0 = x(t_0), \quad y(t) = Cx(t) + Du(t)$$



Relacje pomiędzy opisami

Równania stanu

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad x_0 = x(t_0), \quad y(t) = Cx(t) + Du(t)$$

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$x(t_{n+1}) = e^{A(t_{n+1}-t_n)}x(t_n) + \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-\tau)}Bu^*(\tau)d\tau, \quad u^*(\tau) = u(t_n)\delta(\tau - t_n)$$

$$x(t_{n+1}) = e^{A(t_{n+1}-t_n)}x(t_n) + \int_{t_n}^{t_{n+1}} e^{A(t_{n+1}-\tau)}Bu(t_n)\delta(\tau - t_n)d\tau$$

$$x(t_{n+1}) = e^{A(t_{n+1}-t_n)}x(t_n) + e^{A(t_{n+1}-t_n)}Bu(t_n) = e^{A\Delta t}x(t_n) + e^{A\Delta t}Bu(t_n)$$

niech: $x(t_n) = x_n, \quad u_n = u(t_n), \quad y_n = y(t_n)$

$$\begin{array}{lcl} x_{n+1} = e^{A\Delta t}x_n + e^{A\Delta t}Bu_n & \Rightarrow & \bar{A}x_n + \bar{B}u_n \\ y_n = Cx_n + Du_n & & \bar{C}x_n + \bar{D}u_n \end{array}$$

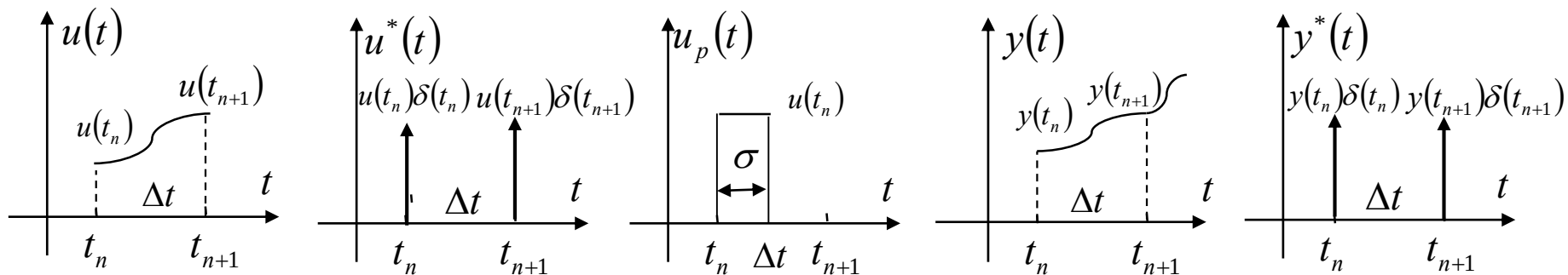
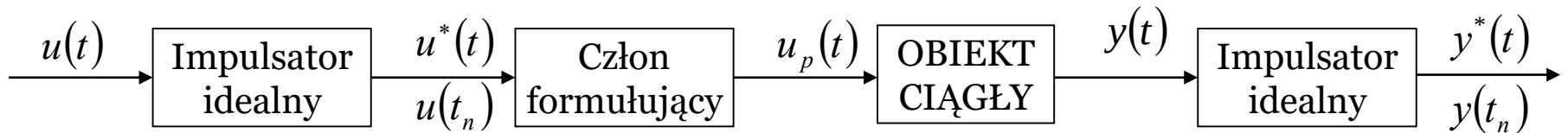
gdzie: $\bar{A} = e^{A\Delta t}, \quad \bar{B} = e^{A\Delta t}B, \quad \bar{C} = C, \quad \bar{D} = D$

Relacje pomiędzy opisami

Równania stanu

Opis przy pomocy równań stanu – impulsator z członem formułującym

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad x_0 = x(t_0), \quad y(t) = Cx(t) + Du(t)$$



Relacje pomiędzy opisami

Równania stanu

$$x(t_{n+1}) = e^{A(t_{n+1}-t_n)}x(t_n) + \int_{t_n}^{t_{n+1}} e^{A(t_n-\tau)}Bu^*(\tau)d\tau, \quad u^*(\tau) = \begin{cases} u(t_n) & \text{dla } t_n \leq \tau \leq t_n + \sigma \\ 0 & \text{dla } t_n + \sigma \leq \tau \leq t_{n+1} \end{cases}$$

$$x(t_{n+1}) = e^{A(t_{n+1}-t_n)}x(t_n) + \int_{t_n}^{t_n+\sigma} e^{A(t_{n+1}-\tau)}Bu(t_n)d\tau = e^{A\Delta t}x(t_n) + \int_{t_n}^{t_n+\sigma} e^{A(t_n+\Delta t-\tau)}Bu(t_n)d\tau$$

$$\text{niech: } \bar{\tau} = \Delta t - \sigma \quad x(t_{n+1}) = e^{A\Delta t}x(t_n) + \int_{\Delta t-\sigma}^{\Delta t} e^{A\bar{\tau}}Bd\bar{\tau} u(t_n)$$

$$\text{niech: } x(t_n) = x_n, \quad u_n = u(t_n), \quad y_n = y(t_n)$$

$$x_{n+1} = e^{A\Delta t}x_n + u_n \int_{\Delta t-\sigma}^{\Delta t} e^{A\bar{\tau}}Bd\bar{\tau} \quad \Rightarrow \quad \begin{aligned} x_{n+1} &= \bar{A}x_n + \bar{B}u_n \\ y_n &= \bar{C}x_n + \bar{D}u_n \end{aligned}$$

$$y_n = Cx_n + Du_n$$

$$\text{gdzie: } \bar{A} = e^{A\Delta t}, \quad \bar{B} = \int_{\Delta t-\sigma}^{\Delta t} e^{A\bar{\tau}}Bd\bar{\tau}, \quad \bar{C} = C, \quad \bar{D} = D$$

Szczególny przypadek: $\sigma = \Delta t$ Ekstrapolator rzędu zerowego - E_0

$$\bar{B} = \int_{\Delta t-\sigma}^{\Delta t} e^{A\bar{\tau}}Bd\bar{\tau} = \int_0^{\Delta t} e^{A\bar{\tau}}Bd\bar{\tau} = A^{-1}[e^{A\Delta t} - I]B$$

Relacje pomiędzy opisami

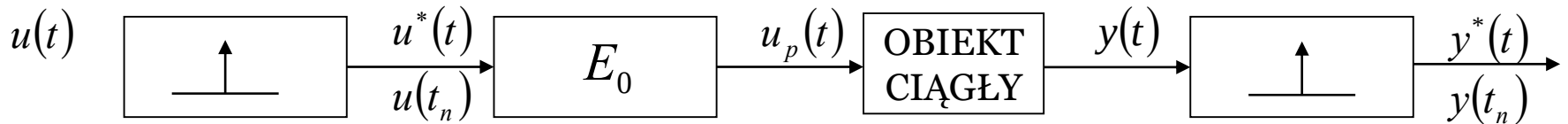
Człon inercyjny pierwszego rzędu

Transmitancja

$$K(s) = \frac{k}{sT + 1}$$

Równania stanu

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t) = -\frac{1}{T}x(t) + \frac{k}{T}u(t) & A = -\frac{1}{T}, \quad B = \frac{k}{T} \\ y(t) &= Cx(t) + Du(t) = x(t) + 0u(t) & C = 1, \quad D = 0 \end{aligned}$$



Dla impulsatora z członem formułującym w postaci ekstrapolatora zerowego rzędu

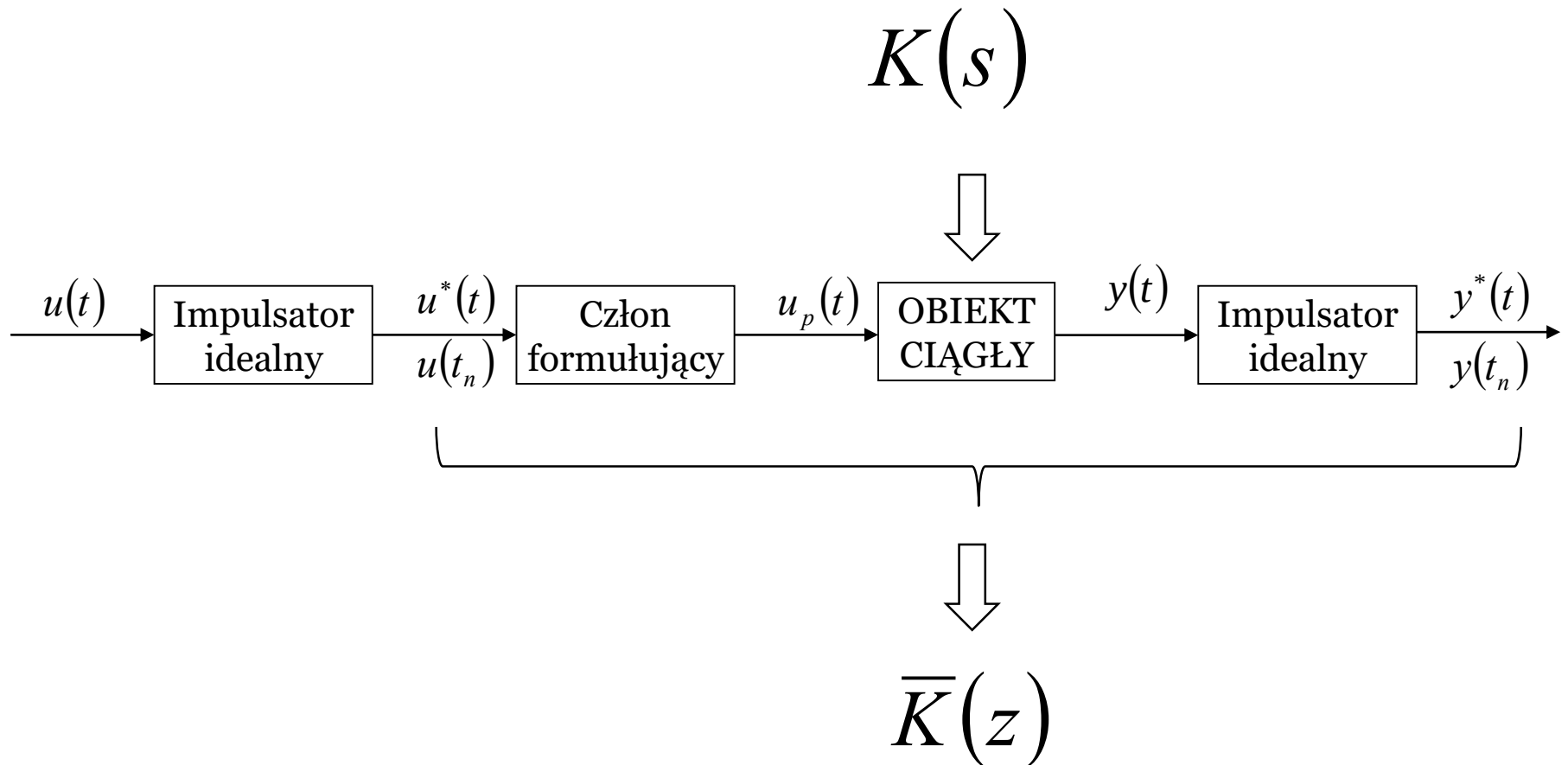
$$\bar{A} = e^{A\Delta t} = e^{-\frac{\Delta t}{T}}, \quad \bar{B} = A^{-1}[e^{A\Delta t} - I]B = -T \left(e^{-\frac{\Delta t}{T}} - 1 \right) \frac{k}{T} = k \left(1 - e^{-\frac{\Delta t}{T}} \right),$$

$$\bar{C} = C = 1, \quad \bar{D} = D = 0$$

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + Bu(t) = -\frac{1}{T}x(t) + \frac{k}{T}u(t) \\ y(t) &= Cx(t) + Du(t) = x(t) + 0u(t) \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_{n+1} &= \bar{A}x_n + \bar{B}u_n = e^{-\frac{\Delta t}{T}}x_n + k \left(1 - e^{-\frac{\Delta t}{T}} \right) u_n \\ y_n &= \bar{C}x_n + \bar{D}u_n = x_n \end{aligned}$$

Relacje pomiędzy opisami

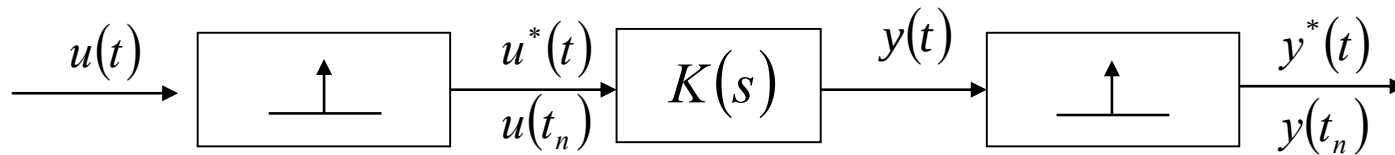
Transmitancja obiektu



Relacje pomiędzy opisami

Transmitancja obiektu

Impulsator idealny (bez członu formułującego)



$$K(s) \Rightarrow k_i(t) = \mathcal{L}^{-1}(K(s)) \Rightarrow k_i(n\Delta t) \Rightarrow \bar{K}(z) = \mathcal{Z}\{k_i(n\Delta t)\}$$

Przykład:

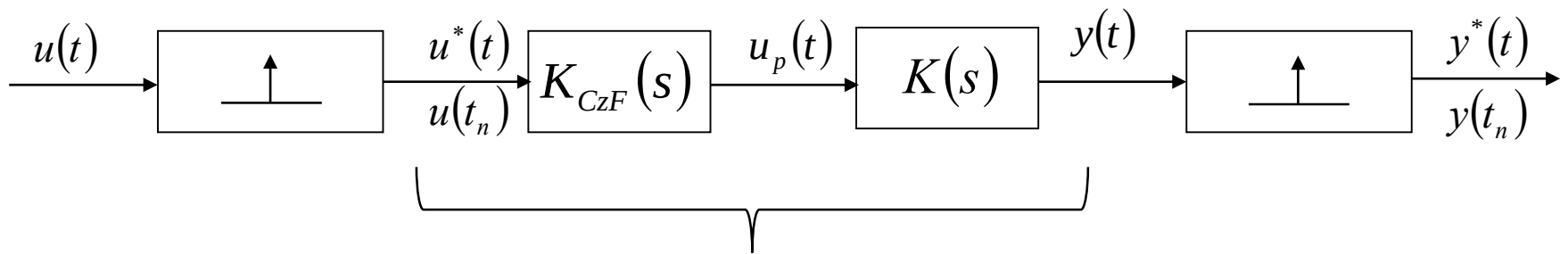
$$K(s) = \frac{k}{1+sT} \Rightarrow k_i(t) = \mathbf{1}(t) \frac{k}{T} e^{-\frac{t}{T}} \Rightarrow k_i(n\Delta t) = \mathbf{1}_n \frac{k}{T} e^{-\frac{n\Delta t}{T}}$$

$$\bar{K}(z) = \mathcal{Z}\{k_i(n\Delta t)\} = \mathcal{Z}\left\{\mathbf{1}_n \frac{k}{T} e^{-\frac{n\Delta t}{T}}\right\} = \frac{k}{T} \frac{z}{z - e^{-\frac{\Delta t}{T}}}$$

Relacje pomiędzy opisami

Transmitancja obiektu

Impulsator z członem formułującym

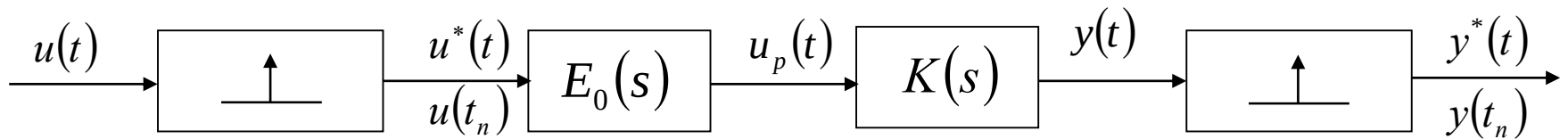


$$K_Z(s) = K_{CzF}(s)K(s)$$

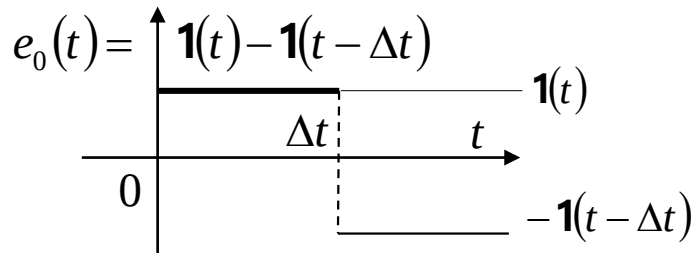
$$K_Z(s) \Rightarrow k_{iZ}(t) = \mathcal{L}^{-1}(K_Z(s)) \Rightarrow k_{iZ}(n\Delta t) \Rightarrow \bar{K}(z) = \mathcal{Z}\{k_{iZ}(n\Delta t)\}$$

Relacje pomiędzy opisami

Transmitancja obiektu - przykład



$$K_{CzF}(s) = E_0(s) = \mathcal{L}(e_0(t)) = \mathcal{L}(\mathbf{1}(t) - \mathbf{1}(t - \Delta t)) = \frac{1 - e^{-s\Delta t}}{s} \quad K(s) = \frac{k}{1 + sT}$$



$$K_Z(s) = K_{CzF}(s)K(s) = \frac{1 - e^{-s\Delta t}}{s} \frac{k}{1 + sT}$$

$$k_{iZ}(t) = \mathcal{L}^{-1}\left(\frac{1 - e^{-s\Delta t}}{s} \frac{k}{1 + sT}\right) = \mathbf{1}(t)k\left(1 - e^{-\frac{t}{T}}\right) - \mathbf{1}(t - \Delta t)k\left(1 - e^{-\frac{t - \Delta t}{T}}\right)$$

$$k_{iZ}(n\Delta t) = \mathbf{1}_n k\left(1 - e^{-\frac{n\Delta t}{T}}\right) - \mathbf{1}_{n-1} k\left(1 - e^{-\frac{n\Delta t - \Delta t}{T}}\right) = \mathbf{1}_n k\left(1 - e^{-\frac{n\Delta t}{T}}\right) - \mathbf{1}_{n-1} k\left(1 - e^{-\frac{(n-1)\Delta t}{T}}\right)$$

Relacje pomiędzy opisami

Transmitancja obiektu - przykład

$$\begin{aligned}
 \bar{K}(z) &= \mathcal{Z}\{k_{iZ}(n\Delta t)\} = \mathcal{Z}\left\{\mathbf{1}_n k\left(1 - e^{-\frac{n\Delta t}{T}}\right) - \mathbf{1}_{n-1} k\left(1 - e^{-\frac{(n-1)\Delta t}{T}}\right)\right\} = \\
 &\mathcal{Z}\left\{\mathbf{1}_n k\left(1 - e^{-\frac{n\Delta t}{T}}\right)\right\} - \frac{1}{z} \mathcal{Z}\left\{\mathbf{1}_n k\left(1 - e^{-\frac{n\Delta t}{T}}\right)\right\} = \\
 &k\left(\frac{z}{z-1} - \frac{z}{z - e^{-\frac{\Delta t}{T}}}\right) - k \frac{1}{z} \left(\frac{z}{z-1} - \frac{z}{z - e^{-\frac{\Delta t}{T}}}\right) = \\
 &k\left(\frac{z-1}{z-1} - \frac{z-1}{z - e^{-\frac{\Delta t}{T}}}\right) = k\left(1 - \frac{z-1}{z - e^{-\frac{\Delta t}{T}}}\right) = k \frac{1 - e^{-\frac{\Delta t}{T}}}{z - e^{-\frac{\Delta t}{T}}}
 \end{aligned}$$

Dziękuję za uwagę

