Computer Science

Jerzy Świątek

Systems Modelling and Analysis

Choose yourself and new technologies

L.20. Decision making under uncertainty

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UNCERTAINTY – RANDOM VARIABLE

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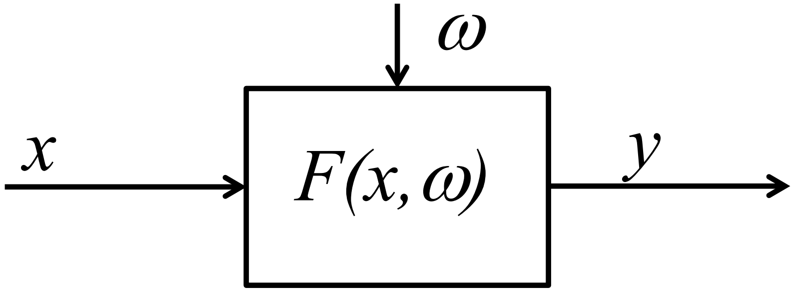
Decision making under uncertainty



$$x^{*} \rightarrow F\left(x^{*},\omega\right) = \min_{x \in D_x(\omega)} F\left(x,\omega\right) \quad ???$$

The choice of variant

Let us consider an example:

The network of I mines to be modernized is given. There are J variants of modernization, each involving costs c_{ij} if i-th mine is modernized by j-th variant ($i = 1, 2, \dots, I, \quad j = 1, 2, \dots, J$). Each mine's spoil is u_{ij} - if i-th mine is modernized by j-th variant. Contamination of each mine's spoil is z_{ij} - if i-th mine is modernized by j-th variant. The task is to assign variants to mines in such a way that total spoil do not fall below *u*, contaminations of spoil meet the requirements of the market (defined by the upper limit *z*) and total costs are as small as possible.

Decision making under uncertainty

$$F(x,c)=\sum_{i=1}^{I}\sum_{j=1}^{J}c_{ij}x_{ij}$$

x_{ij} - decision variable

$$\sum_{i=1}^{I}\sum_{j=1}^{J}u_{ij}x_{ij}\geq u$$

$x_{ij} = \begin{cases} 1 & \text{- } i\text{–th mine modernized by } j\text{–th variant} \\ 0 & \text{- otherwise} \end{cases}$

$$\sum_{j=1}^{J}z_{ij}x_{ij}\leq z \quad i=1,2,\dots,I$$

$$\sum_{j=1}^{J}x_{ij}=1 \quad i=1,2,\dots,I$$

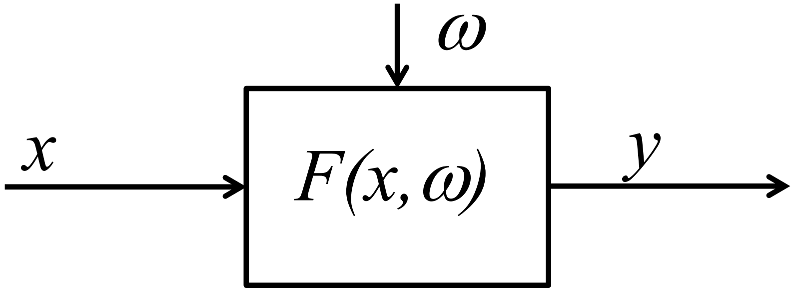
Decision making under uncertainty

$\{\underline{c}_{ij}, \underline{u}_{ij}, \underline{z}_{ij}\ i=1,\dots,I,\ j=1,\dots,J\} = \underline{\omega}$ - uncertain (random) variables

$$F(x,\underline{c}) = \sum_{i=1}^{I}\sum_{j=1}^{J}\underline{c}_{ij}x_{ij} \quad \Rightarrow F(x,\underline{\omega})$$

$$\left.\begin{array}{l} \sum_{i=1}^{I}\sum_{j=1}^{J}\underline{u}_{ij}x_{ij} \ge u \\ \sum_{j=1}^{J}\underline{z}_{ij}x_{ij} \le z \quad i=1,\dots,I \\ \sum_{j=1}^{J}x_{ij} = 1 \quad i=1,2,\dots,I \end{array}\right\} \Rightarrow \mathscr{D}_x(\underline{\omega}) = \{x \in \mathscr{R}^S;\ \varphi_l(x,\underline{\omega}) = 0,\ l=1,\dots,L,\ \psi_m(x,\underline{\omega}) \le 0,\ m=1,\dots,M\}$$

Decision making under uncertainty



$$x^* \rightarrow F\left(x^*, \omega\right) = \min_{x \in D_x(\omega)} F\left(x, \omega\right) \quad ???$$

Decision making under uncertainty

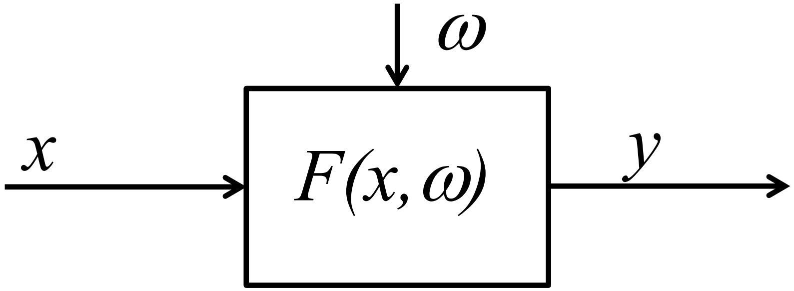
$$\underset{\underline{c}}{E}\left[F\left(x,\underline{c}\right)\right]=\underset{\underline{c}}{E}\left[\sum_{i=1}^{I}\sum_{j=1}^{J}\underline{c}_{ij}x_{ij}\right]$$

$$\underset{\underline{u}}{E}\left[\sum_{i=1}^{I}\sum_{j=1}^{J}\underline{u}_{ij}x_{ij}\right]\geq u$$

$$\underset{\underline{z}}{E}\left[\sum_{j=1}^{J}\underline{z}_{ij}x_{ij}\right]\leq z \quad i=1,2,\ldots,I$$

$$\sum_{j=1}^{J}x_{ij}=1 \quad i=1,2,\ldots,I$$

Decision making under uncertainty



$$F(x) = \underset{\underline{\omega}}{E}\left[F(x,\underline{\omega})\right]$$

$$\mathscr{D}_x = \underset{\underline{\omega}}{E}\left[\mathscr{D}_x(\underline{\omega})\right] =$$

$$= \left\{ x \in \mathscr{R}^S;\ \underset{\underline{\omega}}{E}\left[\varphi_l(x,\underline{\omega})\right] = 0,\ l = 1, \ldots, L,\ \underset{\underline{\omega}}{E}\left[\psi_m(x,\underline{\omega})\right] \le 0,\ m = 1, \ldots, M \right\}$$

$$x^* \rightarrow F(x^*) = \min_{x \in D_x} F(x)$$

Decision making under uncertainty

ω - value of random variable $\underline{\omega}$

$\omega \in \Omega$ - a continuous set,

$f_{\omega}(\omega)$ - probability density functions of random variable $\underline{\omega}$

Then:

$$F(x) = \underset{\underline{\omega}}{E}[F(x,\underline{\omega})] = \int_{\Omega} F(x,\omega) f_{\omega}(\omega) d\omega$$

$$\mathscr{D}_x = \underset{\underline{\omega}}{E}[\mathscr{D}_x(\underline{\omega})] = \Big\{ x \in \mathscr{R}^S ;\ \underset{\underline{\omega}}{E}[\varphi_l(x,\underline{\omega})] = \int_{\Omega} \varphi_l(x,\omega) f_{\omega}(\omega) d\omega = 0,\ l = 1, \ldots, L,$$

$$\underset{\underline{\omega}}{E}[\psi_m(x,\underline{\omega})] = \int_{\Omega} \psi_m(x,\omega) f_{\omega}(\omega) d\omega \le 0,\ m = 1, \ldots, M \Big\}$$

Decision making under uncertainty

ω - value of random variable $\underline{\omega}$

$\omega \in \Omega = \{\omega_1, \omega_2, \ldots, \omega_K\}$ - a discrete set,

$P(\underline{\omega} = \omega_k) = p_k,\ k = 1, 2, \ldots, K$ - probability density functions of random variable $\underline{\omega}$

Then:

$$F(x) = \underset{\underline{\omega}}{E}[F(x, \underline{\omega})] = \sum_{k=1}^{K} F(x, \omega_k) p_k$$

$$\mathscr{D}_x = \underset{\underline{\omega}}{E}[\mathscr{D}_x(\underline{\omega})] = \Big\{ x \in \mathscr{R}^S;\ \underset{\underline{\omega}}{E}[\varphi_l(x, \underline{\omega})] = \sum_{k=1}^{K} \varphi_l(x, \omega_k) p_k = 0,\ l = 1, \ldots, L,$$

$$\underset{\underline{\omega}}{E}[\psi_m(x, \underline{\omega})] = \sum_{k=1}^{K} \psi_m(x, \omega_k) p_k \le 0,\ m = 1, \ldots, M \Big\}$$

Decision making under uncertainty

The problem of newspapers vendor.

A newspaper vendor makes an order for bundles of 40 newspapers. Quantity price of one newspaper is 0.80 euro, and vendors sells it for 1.10 euro. Demand for newspapers $\underline{\omega}$ is random variable. The „bad day" sale $\omega_1 = 50$ newspapers, the „moderate day" sale $\omega_2 = 100$ newspapers, and the „good day" $\omega_3 = 150$ newspapers. Probability of the „bad day" is $P(\underline{\omega} = \omega_1) = p_1 = 0.26$, of the „moderate day" is $P(\underline{\omega} = \omega_2) = p_2 = 0.40$, of the „good day" is $P(\underline{\omega} = \omega_3) = p_3 = 0.34$.

Decision variable *x* is the number of bundles the vendor should order.

x =1, 2, 3, 4, … ?

Decision making under uncertainty

The goal function:

$$F(x,\underline{\omega})=\begin{cases} 40x\times 0.30 & if \quad 40x<\underline{\omega} \\ \underline{\omega}\times 0.30-(40x-\underline{\omega})\times 0.80 & if \quad 40x\geq\underline{\omega} \end{cases}$$

Value of the goal function for x = 1, 2, 3 i 4

x	$\boldsymbol{\omega_1 = 50}$	$\boldsymbol{\omega_2 = 100}$	$\boldsymbol{\omega_3 = 150}$
	$p_1 = 0.26$	$p_2 = 0.40$	$p_3 = 0.34$
x=1	12	12	12
x=2	-9	24	24
x=3	-41	14	36
x=4	-73	-18	37

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Decision making under uncertainty

$$F(x) = \underset{\underline{\omega}}{E}\left[F(x,\underline{\omega})\right] = \sum_{k=1}^{K} F(x,\omega_k)p_k$$

$$F(x=1) = \underset{\underline{\omega}}{E}\left[F(x=1,\underline{\omega})\right] = 12\times 0.26 + 12\times 0.4 + 12\times 0.34 = 12$$

$$F(x=2) = \underset{\underline{\omega}}{E}\left[F(x=2,\underline{\omega})\right] = -9\times 0.26 + 24\times 0.4 + 24\times 0.34 = 15.42$$

$$F(x=3) = \underset{\underline{\omega}}{E}\left[F(x=3,\underline{\omega})\right] = -41\times 0.26 + 14\times 0.4 + 36\times 0.34 = 7.17$$

$$F(x=1) = \underset{\underline{\omega}}{E}\left[F(x=1,\underline{\omega})\right] = -73\times 0.26 - 18\times 0.4 + 37\times 0.34 = -13.6$$

x	$\omega_1 = 50$	$\omega_1 = 100$	$\omega_1 = 150$	*F(x)*
	$p_1 = 0.26$	$p_1 = 0.40$	$p_1 = 0.34$	
x=1	12	12	12	12
x=2	-9	24	24	15.42
x=3	-41	14	36	7.17
x=4	-73	-18	37	-13.6

Decision making under uncertainty

The cautious vendor suppresses profits and exaggerates losses

$$\overline{F}(x,\underline{\omega})=\begin{cases}10\times\sqrt{F(x,\underline{\omega})} & if \quad F(x,\underline{\omega})\geq 0\\ -\dfrac{\left(F(x,\underline{\omega})\right)^2}{10} & if \quad F(x,\underline{\omega})<0\end{cases}$$

x	$\omega_1 = 50$	$\omega_2 = 100$	$\omega_3 = 150$	***F(x)***
	$p_1 = 0.26$	$p_2 = 0.40$	$p_3 = 0.34$	
x=1	34.64	34.64	34.64	34.64
x=2	-8.1	48.99	48.99	34.15
x=3	-168.1	37.42	60	-8.34
x=4	-532.9	-32.4	60.83	-130.83

Decision making under uncertainty

The „risk-taking” vender exaggerates profits and suppresses losses

$$\overline{F}(x,\underline{\omega}) = \begin{cases} \dfrac{\left(F(x,\underline{\omega})\right)^2}{10} & if \quad F(x,\underline{\omega}) \geq 0 \\ -10 \times \sqrt{-F(x,\underline{\omega})} & if \quad F(x,\underline{\omega}) < 0 \end{cases}$$

x	$\omega_1 = 50$ $p_1 = 0.26$	$\omega_2 = 100$ $p_2 = 0.40$	$\omega_3 = 150$ $p_3 = 0.34$	*F(x)*
x=1	14.4	14.4	14.4	14.4
x=2	-30	57.6	57.6	34.82
x=3	-64.03	19.6	129.6	35.26
x=4	-85.44	-42.43	136.9	7.36

A game against nature

A GAME-THEORETIC APPROACH TO DECISION MAKING UNDER UNCERTAINTY

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A game against nature

A farmer examines the possibility of growing 5. different types of corn. The size of each grain yield depends on weather conditions. In terms of humidity year may be dry, normal or wet. The table below presents expected yield for different weather conditions.

Type of corn	Weather conditions		
	drought	normal	rain
1	8	10	12
2	10	11	7
3	9	13	8
4	11	10	6
5	10	10	9

A game against nature

The min – max rule.

Analyzing the subsequent rows of the matrix we find the maximum revenue that may be achieved for successive states of a nature. We make such a decision, for which the maximum revenue is the smallest. In case of ambiguity, we recommend all the decisions for which the above condition is satisfied.

Type of corn	Weather conditions			max
	drought	normal	rain	
1	8	10	12	12
2	10	11	7	11
3	9	13	8	13
4	11	10	6	11
5	10	10	9	10

← min

A game against nature

The WALD's (max – min) rule.
Analyzing the subsequent rows of the matrix we find the minimum revenue that may be achieved for successive states of a nature. We make such a decision, for which the minimum revenue is the greatest. In case of ambiguity, we recommend all the decisions for which the above condition is satisfied.

Type of corn	Weather conditions			min	
	drought	normal	rain		
1	8	10	12	8	
2	10	11	7	7	
3	9	13	8	8	
4	11	10	6	6	
5	10	10	9	9	← max

A game against nature

The max – max rule.

Analyzing the subsequent rows of the matrix we find the maximum revenue that may be achieved for successive states of a nature. We make such a decision, for which the maximum revenue is the greatest. In case of ambiguity, we recommend all the decisions for which the above condition is satisfied.

Type of corn	Weather conditions			max	
	drought	normal	rain		
1	8	10	12	12	
2	10	11	7	11	
3	9	13	8	13	← max
4	11	10	6	11	
5	10	10	9	10	

A game against nature

a_i - the minimum profit for *i*–th row

A_i - the maximum profit for *i*–th row

γ - confidence factor

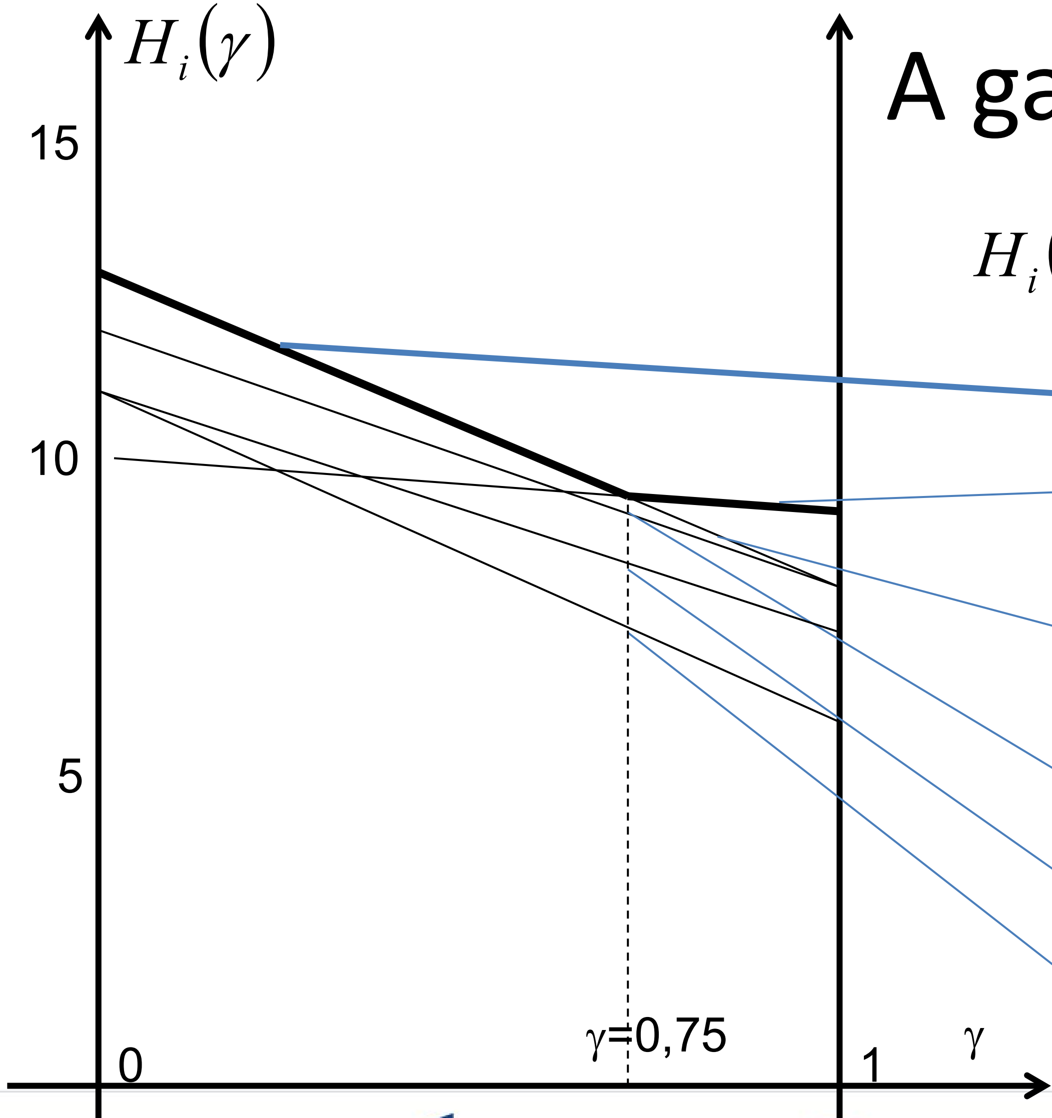
$$H_i(\gamma) = a_i\,\gamma + A_i(1-\gamma) \quad \gamma \in [0,1]$$

The Hurwitz rule.

Analyzing the subsequent rows of the matrix we find the minimum and the maximum revenue, i.e. values a_i, A_i and value of the function $H_i(\gamma)$ for a given γ. We make such a decision, for which the value of the function $H_i(\gamma)$ is the greatest. In case of ambiguity, we recommend all the decisions for which the above condition is satisfied.

Type of corn	Weather conditions			*a* min	*A* max	$H(\gamma)$ $\gamma = 0.5$
	drought	normal	rain			
1	8	10	12	8	12	10
2	10	11	7	7	11	9
3	9	13	8	8	13	10.5 ← max
4	11	10	6	6	11	8.5
5	10	10	γ 9	9	10	9.5

A game against nature



$$H_i(\gamma) = a_i\,\gamma + A_i(1-\gamma) \quad \gamma \in [0, 1]$$

$$H(\gamma) = \max_{1 \le i \le 5} \{H_i(\gamma)\}$$

$$H_5(\gamma) = 9\,\gamma + 10(1-\gamma)$$

$$H_3(\gamma) = 8\,\gamma + 13(1-\gamma)$$

$$H_1(\gamma) = 8\,\gamma + 12(1-\gamma)$$

$$H_2(\gamma) = 7\,\gamma + 11(1-\gamma)$$

$$H_4(\gamma) = 6\,\gamma + 11(1-\gamma)$$

A game against nature

The Laplace rule.

Analyzing the subsequent rows of the matrix we find the expected revenue, assuming that successive states of nature are equally likely. We make such a decision, for which expected revenue is the greatest. In case of ambiguity, we recommend all the decisions for which the above condition is satisfied.

Type of corn	Weather conditions			Expected revenue	
	drought	normal	rain		
1	8	10	12	30/3	← max
2	10	11	7	28/3	
3	9	13	8	30/3	← max
4	11	10	6	27/3	
5	10	10	9	19/3	

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Two-person zero-sum game

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Two-person zero-sum game

Two-player zero-sum game

Payoff matrix of player A:

B \ A	B_1	B_2	…	B_m	…	B_M
A_1	a_{11}	a_{12}	…	a_{1m}	…	a_{1M}
A_2	a_{21}	a_{22}	…	a_{2m}	…	a_{2M}
…	…	…	…	…	…	…
A_n	a_{n1}	a_{n2}	…	a_{nm}	…	a_{nM}
…	…	…	…	…	…	…
A_N	a_{N1}	a_{N2}	…	a_{Nm}	…	a_{NM}

Payoff matrix of player B:

B \ A	B_1	B_2	…	B_m	…	B_M
A_1	$-a_{11}$	$-a_{12}$	…	$-a_{1m}$	…	$-a_{1M}$
A_2	$-a_{21}$	$-a_{22}$	…	$-a_{2m}$	…	$-a_{2M}$
…	…	…	…	…	…	…
A_n	$-a_{n1}$	$-a_{n2}$	…	$-a_{nm}$	…	$-a_{nM}$
…	…	…	…	…	…	…
A_N	$-a_{N1}$	$-a_{N2}$	…	$-a_{Nm}$	…	$-a_{NM}$

Player A aims to maximize revenue

Player B aims to minimize losses

Usually the payoff matrix of player A is presented

Example

Two candidates A and B compete for a parliamentary seat in an electoral district. They have to make decision concerning to carry election campaign in the last weekend before the election. Each of candidates can spend one day in the city M_1 or M_2. They consider (independently) three possible strategies :

A_1, B_1 – to spend one day in both cities M_1 i M_2,

A_2, B_2 – to spend two days in M_1,

A_3, B_3 – to spend two days in M_2.

If candidate A chooses strategy A_1, and candidate B – accordingly chooses strategies B_1, B_2 or B_3, then candidate A may expect gain of votes by 1%, 2% or 4%.

If candidate A chooses strategy A_2, and candidate B – accordingly chooses strategies B_1, B_2 or B_3, then candidate A may expect gain of votes by 1%, 0% or 5%.

If candidate A chooses strategy A_3, and candidate B – accordingly chooses strategies B_1, B_2 or B_3, then candidate A may expect gain of votes by 0%, 1% or decrease by 1%.

Example of payoff matrix

Two-player zero-sum game
Payoff matrix of player A:

A \ B		B_1	B_2	B_3
	A_1	1	2	4
	A_2	1	0	5
	A_3	0	1	-1

Payoff matrix of player B:

A \ B		B_1	B_2	B_3
	A_1	-1	-2	-4
	A_2	-1	0	-5
	A_3	0	-1	1

Player A obtains revenue at the expense of player A and vice versa, hence the sum of payoff matrices of players A and B is zero matrix.

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Two-person zero-sum game

Two-player zero-sum game

Payoff matrix of player A:

B / A	B_1	B_2	…	B_m	…	B_M
A_1	a_{11}	a_{12}	…	a_{1m}	…	a_{1M}
A_2	a_{21}	a_{22}	…	a_{2m}	…	a_{2M}
…	…	…	…	…	…	…
A_n	a_{n1}	a_{n2}	…	a_{nm}	…	a_{nM}
…	…	…	…	…	…	…
A_N	a_{N1}	a_{N2}	…	a_{Nm}	…	a_{NM}

Payoff matrix of player B:

B / A	B_1	B_2	…	B_m	…	B_M
A_1	$-a_{11}$	$-a_{12}$	…	$-a_{1m}$	…	$-a_{1M}$
A_2	$-a_{21}$	$-a_{22}$	…	$-a_{2m}$	…	$-a_{2M}$
…	…	…	…	…	…	…
A_n	$-a_{n1}$	$-a_{n2}$	…	$-a_{nm}$	…	$-a_{nM}$
…	…	…	…	…	…	…
A_N	$-a_{N1}$	$-a_{N2}$	…	$-a_{Nm}$	…	$-a_{NM}$

Player A aims to maximize revenue

Player B aims to minimize losses

Usually the payoff matrix of player A is presented

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Decision making using game theory

- Typical approaches to game solving
  - determination of saddle point
  - removal of dominated strategies
  - determination of mixed strategies for:
    - N=2 and M=2
    - N>2 and M>2

Decision making using game theory

Two-person zero-sum game

Saddle point: $\max_{1 \le n \le N} \min_{1 \le m \le M} a_{nm} \neq ?= \min_{1 \le n \le N} \max_{1 \le m \le M} a_{nm}$

A \ B		B_1	B_2	…	B_m	…	B_M	min
	A_1	a_{11}	a_{12}	…	a_{1m}	…	a_{1M}	
	A_2	a_{21}	a_{22}	…	a_{2m}	…	a_{2M}	
	…	…	…	…	…	…	…	
	A_n	a_{n1}	a_{n2}	…	a_{nm}	…	a_{nM}	
	…	…	…	…	…	…	…	
	A_N	a_{N1}	a_{N2}	…	a_{Nm}	…	a_{NM}	
max								

$\leftarrow \max_{1 \le n \le N} \min_{1 \le m \le M} a_{nm}$

$\uparrow \min_{1 \le n \le N} \max_{1 \le m \le M} a_{nm}$

Decision making using game theory

Two-person zero-sum game

Saddle point: $\max_{1\le n\le N}\min_{1\le m\le M} a_{nm} = \min_{1\le n\le N}\max_{1\le m\le M} a_{nm}$ = 190

B / A		B_1	B_2	B_3	B_4	B_5	min
	A_1	180	150	230	170	150	150
	A_2	200	210	220	150	190	150
	A_3	210	230	190	190	200	190
	A_4	150	220	170	180	220	150
	A_5	210	200	160	150	210	150
max		210	230	230	190	220	

← $\max_{1\le n\le N}\min_{1\le m\le M} a_{nm}$ (row A_3)

↑ $\min_{1\le n\le N}\max_{1\le m\le M} a_{nm}$ (column B_4)

Decision making using game theory

Dominant and dominated strategies

Player *A* may choose between strategies: $A_1, A_2, \ldots, A_N$

A strategy $A_{n'}$ is dominated by a dominant strategy $A_{n''}$ if

$$\forall\ m = 1, 2, \ldots M \quad a_{n'm} \leq a_{n''m}$$

Player *B* may choose between strategies: $B_1, B_2, \ldots, B_M$

A strategy $B_{m'}$ is dominated by a dominant strategy $B_{m''}$ if

$$\forall\ n = 1, 2, \ldots N \quad a_{nm'} \geq a_{nm''}$$

Example

Removal of dominated strategies

Step 1.

A \ B	B_1	B_2	B_3	
A_1	1	2	4	← dominant strategy
A_2	1	0	5	
~~A_3~~	~~0~~	~~1~~	~~-1~~	← dominated strategy

Step 2.

A \ B	B_1	B_2	~~B_3~~
A_1	1	2	~~4~~
A_2	1	0	~~5~~

↑ - dominated strategy

↑

↑ - dominant strategy

Example

Removal of dominated strategies

Step 3.

A \ B			
		B_1	B_2
	A_1	1	2
	~~A_2~~	~~1~~	~~0~~

← dominant strategy (row A_1)

← dominated strategy (row A_2)

Step 4.

A \ B			
		B_1	~~B_2~~
	A_1	1	~~2~~

↑ - dominated strategy (B_2)

↑ - dominant strategy (B_1)

Result of the game (→ 1)

Decision making using game theory

Two-player zero-sum game

Mixed strategies : $\max_{1\le n\le N} \min_{1\le m\le M} a_{nm} \neq \min_{1\le n\le N} \max_{1\le m\le M} a_{nm}$

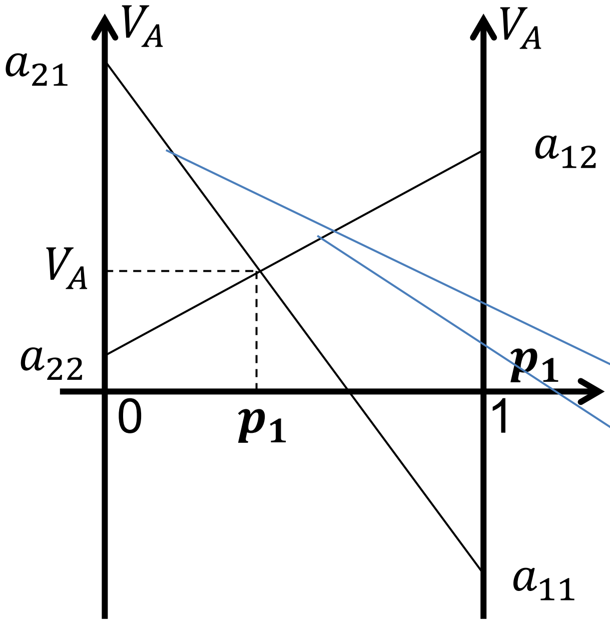
:

B \ A		q_1	q_2	…	q_m	…	q_M
		B_1	B_2	…	B_m	…	B_M
p_1	A_1	a_{11}	a_{12}	…	a_{1m}	…	a_{1M}
p_2	A_2	a_{21}	a_{22}	…	a_{2m}	…	a_{2M}
…	…	…	…	…	…	…	…
p_n	A_n	a_{n1}	a_{n2}	…	a_{nm}	…	a_{nM}
…	…	…	…	…	…	…	…
p_N	A_N	a_{N1}	a_{N2}	…	a_{Nm}	…	a_{NM}

Decision making using game theory

$N = M = 2$

A \ B		q_1	q_2
		B_1	B_2
p_1	A_1	a_{11}	a_{12}
p_2	A_2	a_{21}	a_{22}



Equations for player *A*

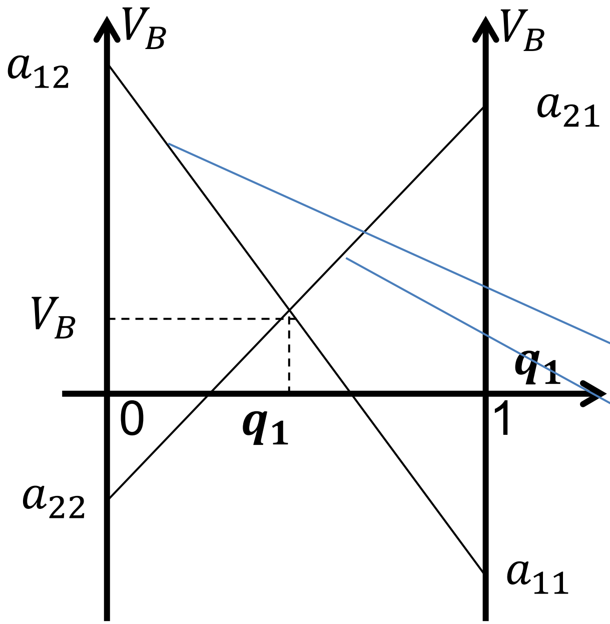
$$p_1 a_{11} + p_2 a_{21} = V_A / B_1$$
$$p_1 a_{12} + p_2 a_{22} = V_A / B_2$$
$$p_1 + p_2 = 1$$

$$p_2 = 1 - p_1$$
$$p_1 a_{11} + (1 - p_1) a_{21} = V_A / B_1$$
$$p_1 a_{12} + (1 - p_1) a_{22} = V_A / B_2$$

Decision making using game theory

$N = M = 2$

A \ B		q_1	q_2
		B_1	B_2
p_1	A_1	a_{11}	a_{12}
p_2	A_2	a_{21}	a_{22}



Equations for player *B*

$$q_1 a_{11} + q_2 a_{12} = V_B \ / \ A_1$$
$$q_1 a_{21} + q_2 a_{22} = V_B \ / \ A_2$$
$$q_1 + q_2 = 1$$

$$q_2 = 1 - q_1$$
$$q_1 a_{11} + (1 - q_1) a_{12} = V_B \ / \ A_1$$
$$q_1 a_{21} + (1 - q_1) a_{22} = V_B \ / \ A_2$$

Example

Two-person zero-sum game

Saddle point: $\max_{1\le n\le N} \min_{1\le m\le M} a_{nm} \neq \min_{1\le n\le N} \max_{1\le m\le M} a_{nm}$

A \ B		B_1	B_2	B_3	min
	A_1	3	-3	7	-3
	A_2	-1	5	2	-1
	A_3	0	-4	4	-4
max		3	5	7	

$\leftarrow \max_{1\le n\le N} \min_{1\le m\le M} a_{nm}$

$\uparrow \min_{1\le n\le N} \max_{1\le m\le M} a_{nm}$

Example

Removal of dominated strategies

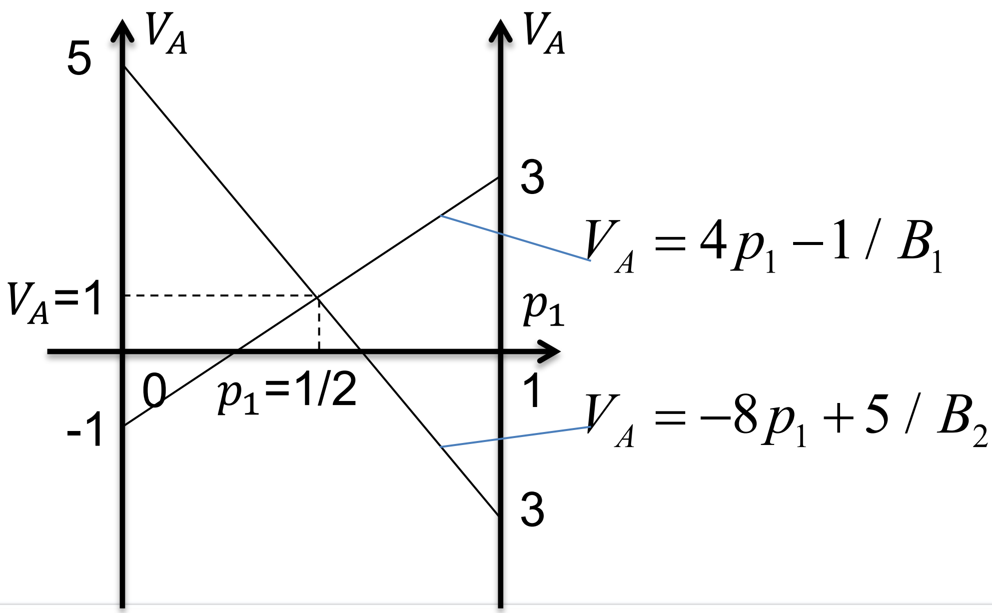
A \ B		q_1	q_2	q_3	
		B_1	B_2	B_3	
p_1	A_1	3	-3	7	← dominant strategy
p_2	A_2	-1	5	2	
p_3	A_3	0	-4	4	← dominated strategy
		↑ - dominant strategy		↑ - dominated strategy	

Since strategies A_3 and B_3 are dominated $\boldsymbol{p_3}$=$\boldsymbol{q_3}$=0, we need to calculate: $\boldsymbol{p_1}$, $\boldsymbol{p_2}$, $\boldsymbol{q_1}$ and $\boldsymbol{q_2}$

Example

$N = M = 2$

A \ B		q_1	q_2
		B_1	B_2
p_1	A_1	3	-3
p_2	A_2	-1	5



Equations for player *A*

$$3p_1 - p_2 = V_A \,/\, B_1$$

$$-3p_1 + 5p_2 = V_A \,/\, B_2$$

$$p_1 + p_2 = 1$$

$$p_2 = 1 - p_1$$

$$3p_1 - (1 - p_1) = V_A \,/\, B_1$$

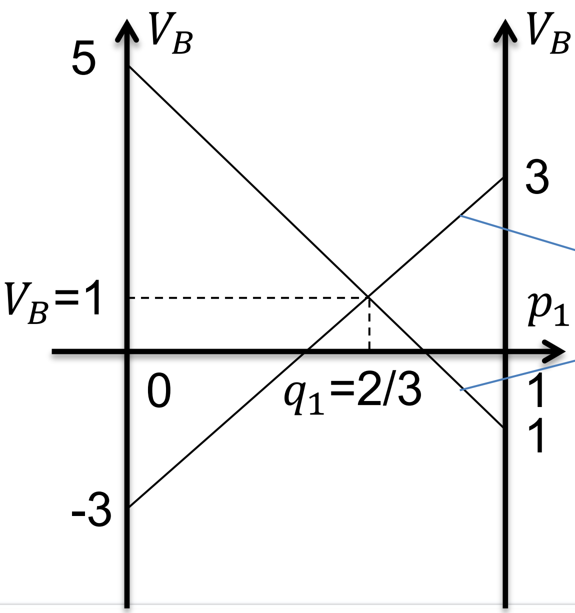
$$-3p_1 + 5(1 - p_1) = V_A \,/\, B_2$$

$$p_1 = \frac{1}{2}, \quad p_2 = \frac{1}{2}$$

$N = M = 2$

Example

A \ B		q_1	q_2
		B_1	B_2
p_1	A_1	3	−3
p_2	A_2	−1	5



$$V_B = 6q_1 - 3 \ / \ A_1$$

$$V_B = -6q_1 + 5 \ / \ A_2$$

Equations for player *B*

$$3q_1 - 3q_2 = V_B \ / \ A_1$$

$$-q_1 + 5q_2 = V_B \ / \ A_2$$

$$q_1 + q_2 = 1$$

$$q_2 = 1 - q_1$$

$$3q_1 - 3(1 - q_1) = V_B \ / \ A_1$$

$$-q_1 + 5(1 - q_1) = V_B \ / \ A_2$$

$$q_1 = \frac{2}{3}, \quad q_2 = \frac{1}{3}$$

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Decision making using game theory

Two-player zero-sum game

Mixed strategies *N*>2, *M*>2:

A \ B		q_1	q_2		q_m		q_M
		B_1	B_2	…	B_m	…	B_M
p_1	A_1	a_{11}	a_{12}	…	a_{1m}	…	a_{1M}
p_2	A_2	a_{21}	a_{22}	…	a_{2m}	…	a_{2M}
…	…	…	…	…	…	…	…
p_n	A_n	a_{n1}	a_{n2}	…	a_{nm}	…	a_{nM}
…	…	…	…	…	…	…	…
p_N	A_N	a_{N1}	a_{N2}	…	a_{Nm}	…	a_{NM}

In this case solving the game reduces to the linear programming task

Decision making using game theory

The task for player A:

$$\left.\begin{array}{l} p_1 a_{1m} + p_2 a_{2m} + \cdots + p_N a_{Nm} \geq V_A \,/\, B_m \quad m = 1, 2, \ldots, M \\ p_1 + p_2 + \cdots + p_N = 1 \\ p_n \geq 0 \quad n = 1, 2, \ldots, N \end{array} \right| \quad /V_A$$

$$\frac{p_1}{V_A} a_{1m} + \frac{p_2}{V_A} a_{2m} + \cdots + \frac{p_N}{V_A} a_{Nm} \geq 1 \,/\, B_m \quad m = 1, 2, \ldots, M$$

$$\frac{p_1}{V_A} + \frac{p_2}{V_A} + \cdots + \frac{p_N}{V_A} = \frac{1}{V_A}$$

$$\frac{p_n}{V_A} \geq 0 \quad n = 1, 2, \ldots, N$$

Let: $x_n = \dfrac{p_n}{V_A} \quad n = 1, 2, \ldots, N$

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Decision making using game theory

The task for player A

$$a_{1m}x_1 + a_{2m}x_2 + \cdots + a_{Nm}x_N \geq 1/B_m \quad m = 1, 2, \ldots, M$$

$$x_1 + x_2 + \cdots + x_N = \frac{1}{V_A}$$

$$x_n \geq 0 \quad n = 1, 2, \ldots, N$$

Therefore, the expression should be minimized

Player A aims to maximize the profit

Finally, the task for player A is:

$$\min_{x_1, x_2, \ldots, x_N} \left(x_1 + x_2 + \cdots + x_N\right) = V_{A\min}$$

with constraints:

$$a_{1m}x_1 + a_{2m}x_2 + \cdots + a_{Nm}x_N \geq 1 \quad m = 1, 2, \ldots, M$$

$$x_n \geq 0 \quad n = 1, 2, \ldots, N$$

Finally, the solution for player A is: $p_n = x_n V_{A\min},\ n = 1, 2, \ldots, N$

Decision making using game theory

The task for player B:

$$\left.\begin{aligned} & q_1 a_{n1} + q_2 a_{n2} + \cdots + q_M a_{nM} \le V_B \;/\; A_n \quad n = 1, 2, \ldots, N \\ & q_1 + q_2 + \cdots + q_M = 1 \\ & q_m \ge 0 \quad m = 1, 2, \ldots, M \end{aligned}\right| \quad /V_B$$

$$\frac{q_1}{V_B} a_{n1} + \frac{q_2}{V_B} a_{n2} + \cdots + \frac{q_M}{V_B} a_{nM} \le 1 \;/\; A_n \quad n = 1, 2, \ldots, N$$

$$\frac{q_1}{V_B} + \frac{q_2}{V_B} + \cdots + \frac{q_M}{V_B} = \frac{1}{V_B}$$

$$\frac{q_m}{V_B} \ge 0 \quad m = 1, 2, \ldots, M$$

Let: $y_m = \frac{q_m}{V_B} \quad m = 1, 2, \ldots, M$

Decision making using game theory

The task for player B

$$a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nM}y_M \le 1 / A_n \quad n = 1, 2, \ldots, N$$

$$y_1 + y_2 + \cdots + y_M = \frac{1}{V_B}$$

$$y_m \ge 0 \quad m = 1, 2, \ldots, M$$

Therefore, the expression should be maximized

Player B minimizes loss

Finally, the task for player B is: $\max\limits_{y_1, y_2, \ldots, y_M} \left(y_1 + y_2 + \cdots + y_M \right) = V_{B\max}$

with constraints: $a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nM}y_M \le 1 \quad n = 1, 2, \ldots, N$

$$y_m \ge 0 \quad m = 1, 2, \ldots, M$$

Finally, the solution for player B is: $q_m = y_n V_{B\max}, \ m = 1, 2, \ldots, M$

Example

Two-person zero-sum game

Saddle point: $\max_{1 \le n \le N} \min_{1 \le m \le M} a_{nm} \neq \min_{1 \le n \le N} \max_{1 \le m \le M} a_{nm}$

A \ B		B_1	B_2	B_3	min
	A_1	5	0	1	0
	A_2	2	4	3	2
max		5	4	3	

$\leftarrow \max_{1 \le n \le N} \min_{1 \le m \le M} a_{nm}$

$\uparrow \min_{1 \le n \le N} \max_{1 \le m \le M} a_{nm}$

Example

B / A		q_1	q_2	q_3
		B_1	B_2	B_3
p_1	A_1	5	0	1
p_2	A_2	2	4	3

The task for player A

$$let\ x_1 = \frac{p_n}{V_A},\ n = 1,2$$

$$\min_{x_1, x_2}(x_1 + x_2)$$

Constraints:

$$5x_1 + 2x_2 \geq 1$$
$$0x_1 + 4x_2 \geq 1$$
$$x_1 + 3x_2 \geq 1$$
$$x_1 \geq 0 \quad x_2 \geq 0$$

The task for player B

$$let\ y_m = \frac{q_m}{V_B},\ m = 1,2,3$$

$$\max_{y_1, y_2, y_3}(y_1 + y_2 + y_3)$$

Constraints:

$$5y_1 + 0y_2 + y_3 \leq 1$$
$$2y_1 + 4y_2 + 3y_3 \leq 1$$
$$y_1 \geq 0\ y_2 \geq 0\ y_3 \geq 0$$

Thank you for attention