Computer Science

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Systems Modelling and Analysis

Choose yourself and new technologies

L.18b. Integer programming

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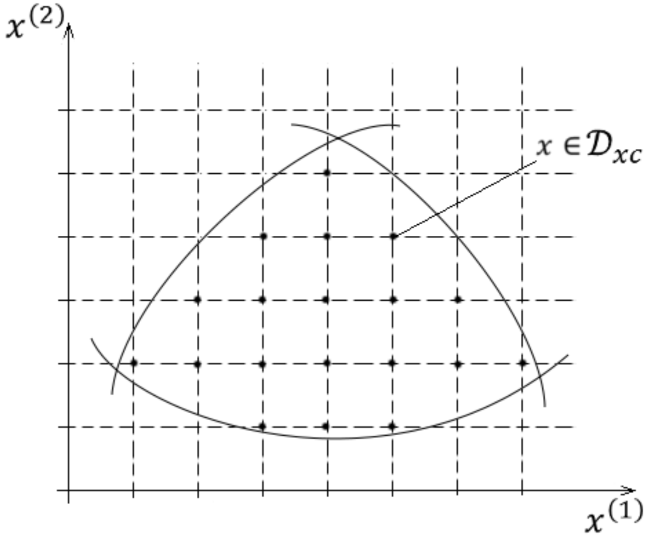
Integer programming – branch and bound method

$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_{xc}} F(x)$$

$\mathcal{D}_{xc} = \mathcal{D}_x \cap \{x^{(s)} \subset C \quad s = 1, 2, \dots, S\}$ integer decision variables

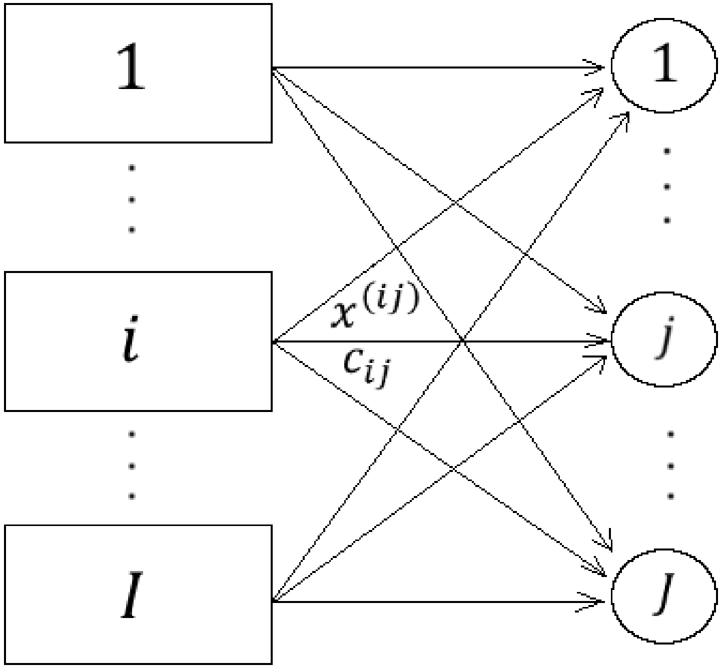


Special case

$\mathcal{D}_{xc} = \{x_1, x_2, \dots, x_k\}$ – finite set, k – large number

$\mathcal{D}_{xc} = \{0, \ 1\}$ - binary programming

Classic problems
Transportation problem



I – the number of suppliers

J – the number of recipients

$x^{(ij)}$ – the number of units of goods delivered by i – th supplier to j – th receiver

c_{ij} – the cost of transport from i – th supplier to j – th receiver

a_i – amount of goods belonging to i – th supplier

b_j – demand for goods of j – th receiver

$i = 1, 2, \dots, I$ $\qquad$ $j = 1, 2, \dots, J$

The goal function – the total cost of transport:

$$F(x) = \sum_{i=1}^{I} \sum_{j=1}^{J} c_{ij}\, x^{(ij)}$$

Constraints:

$$\sum_{j=1}^{J} x^{(ij)} \le a_i \quad i = 1, 2, \dots, I$$

$$\sum_{i=1}^{I} x^{(ij)} = b_j \quad j = 1, 2, \dots, J$$

$$x^{(ij)} \ge 0 \qquad x^{(ij)} \in \mathcal{C}^{+} \quad i = 1, 2, \dots, I$$

$$(w) \quad j = 1, 2, \dots, J$$

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The choice of variant

The set of I facilities is given and J methods of modernization are available. The choice of *j*-th method to modernize *i*-th facility involves costs c_{ij} $(i = 1, 2, \dots, I,$ $j = 1, 2, \dots, J)$. Revenues during modernization are d_{ij}. The problem is to assign methods of modernization to facilities in such a way, that total revenue do not fall below d, and the total cost of modernization is minimal.

Decision variables

$$x^{(ij)} = \begin{cases} 1 & i\text{-th facility is modernized using } j\text{-th method} \\ 0 & \text{otherwise} \end{cases}$$

The goal function

$$F(x) = \sum_{i=1}^{I} \sum_{j=1}^{J} c_{ij} x^{(ij)}$$

Constraints

$$\sum_{i=1}^{I} \sum_{j=1}^{J} d_{ij} x^{(ij)} \geq d$$

$$\sum_{i=1}^{I} x^{(ij)} = 1 \quad i = 1, 2, \dots, I$$

only one method may be chosen to modernize i-th facility

$$x^{(ij)} \in \{0,1\} \quad i = 1, 2, \dots, I, j = 1, 2, \dots, J$$

Knapsack problem

Given a set of items, each with a mass and a value, determine the number of each item to include in a collection so that the total weight is less than or equal to a given limit and the total value is as large as possible.

Notation:

c_s– value of s-th item

d_s – mass of s-th item

d – permissible mass

The goal function:

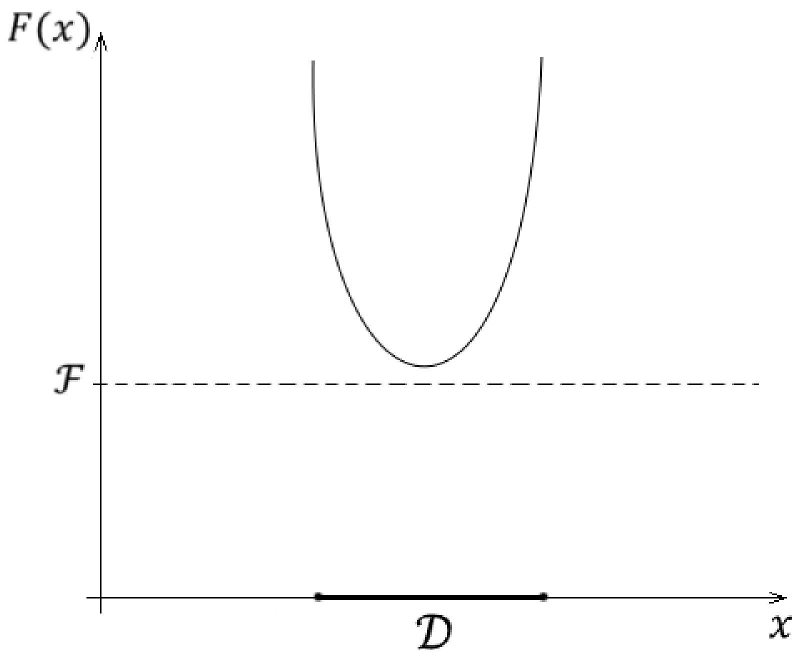
$$F(x) = \sum_{s=1}^{S} c_s x^{(s)}, \; x^{(s)} = \begin{cases} 1 & s\text{-th item} \\ 0 & \text{otherwise} \end{cases}$$

Constraints:

$$\sum_{s=1}^{S} d_s x^{(s)} \leq d$$

Branch and bound method

$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_{xc}} F(x)$$



$\mathcal{F}(\mathcal{D})$ – estimation of the function F on the set $\mathcal{D}$

$$\mathcal{F}(\mathcal{D}) = \inf_{x \in \mathcal{D}} F(x)$$

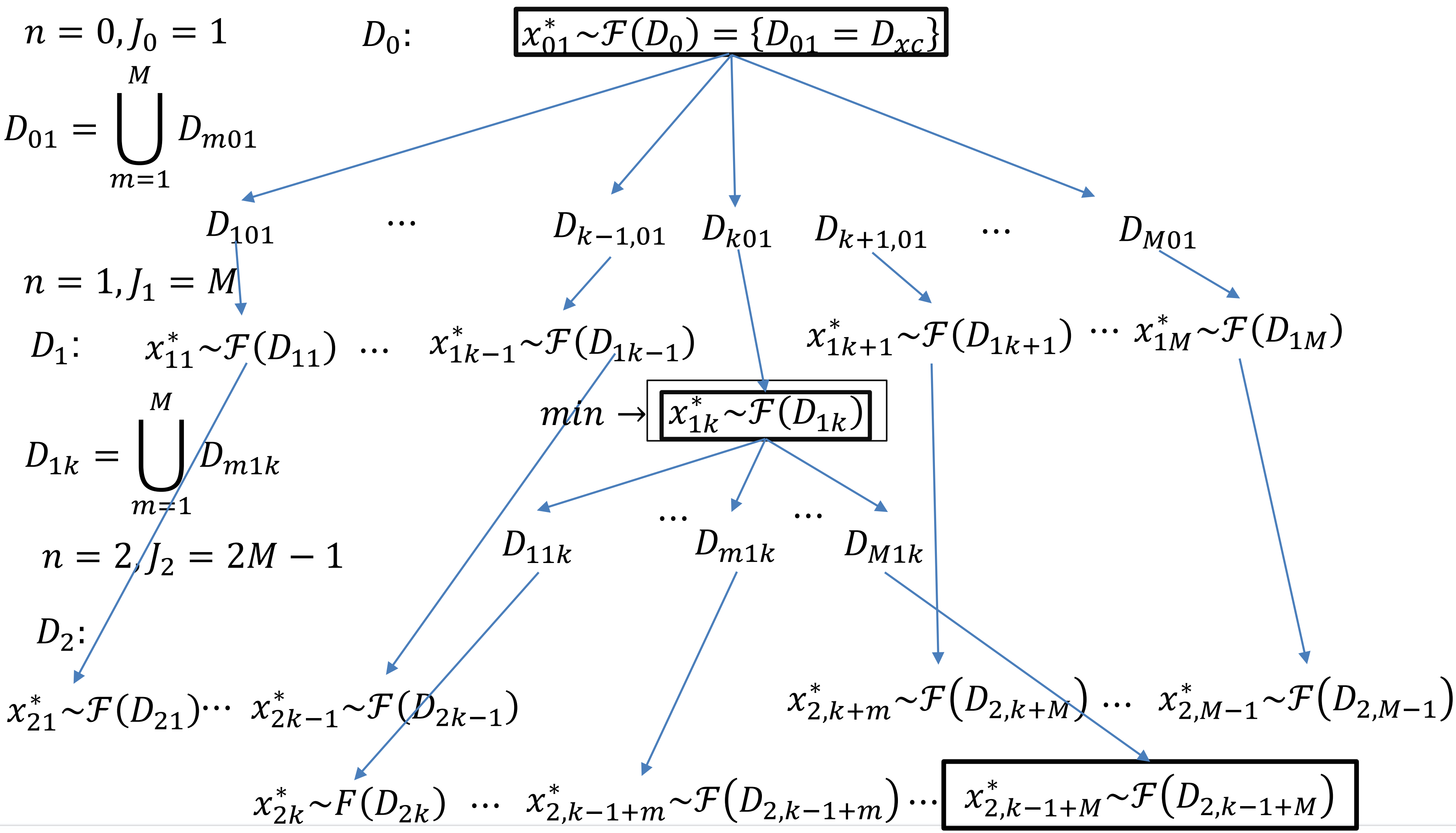
Properties:

$\mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{D}$

$\mathcal{F}(\mathcal{D}_1) \geq \mathcal{F}(\mathcal{D}_2)$

$\mathcal{F}(\{x\}) = F(x)$

$\mathcal{F}(\emptyset) = \infty$


$n = 0, J_0 = 1$
D_0:
$x^*_{01} \sim \mathcal{F}(D_0) = \{D_{01} = D_{xc}\}$
$D_{01} = \bigcup_{m=1}^{M} D_{m01}$
D_{101} … $D_{k-1,01}$ D_{k01} $D_{k+1,01}$ … D_{M01}
$n = 1, J_1 = M$
D_1:
$x^*_{11} \sim \mathcal{F}(D_{11})$ … $x^*_{1k-1} \sim \mathcal{F}(D_{1k-1})$
$x^*_{1k+1} \sim \mathcal{F}(D_{1k+1})$ … $x^*_{1M} \sim \mathcal{F}(D_{1M})$
$min \rightarrow$ $x^*_{1k} \sim \mathcal{F}(D_{1k})$
$D_{1k} = \bigcup_{m=1}^{M} D_{m1k}$
D_{11k} … D_{m1k} … D_{M1k}
$n = 2, J_2 = 2M - 1$
D_2:
$x^*_{21} \sim \mathcal{F}(D_{21})$ … $x^*_{2k-1} \sim \mathcal{F}(D_{2k-1})$
$x^*_{2,k+m} \sim \mathcal{F}(D_{2,k+M})$ … $x^*_{2,M-1} \sim \mathcal{F}(D_{2,M-1})$
$x^*_{2k} \sim F(D_{2k})$ … $x^*_{2,k-1+m} \sim \mathcal{F}(D_{2,k-1+m})$ … $x^*_{2,k-1+M} \sim \mathcal{F}(D_{2,k-1+M})$

Step 0: $\mathcal{D}_0 = \{\mathcal{D}_{xc} = \mathcal{D}_{01}\}, n = 0, J_0 = 1$

Step 1: Determine a set $\mathcal{D}^* \in \mathcal{D}_n$

$$\mathcal{F}(\mathcal{D}^*) = \min_{\mathcal{D} \in \mathcal{D}_n} \mathcal{F}(\mathcal{D})$$

Step 2: Checking whether $\mathcal{D}^*$ is a set ? ($\{x^*\} = \mathcal{D}^*$)
or $x^* \sim \mathcal{F}(\mathcal{D}^*)$ i.e. $\mathcal{F}(\mathcal{D}^*) = F(x^*)$ $x^* \in \mathcal{D}^*$ (?) then x^* optimal solution
STOP

Step 3: $\mathcal{D}^* = \mathcal{D}_{nk}$ is split up into M disjoint sets

$$\mathcal{D}_{1nk} \mathcal{D}_{2nk} \dots \mathcal{D}_{Mnk} \qquad \mathcal{D}_{nk} = \bigcup_{m=1}^{M} \mathcal{D}_{mnk}$$

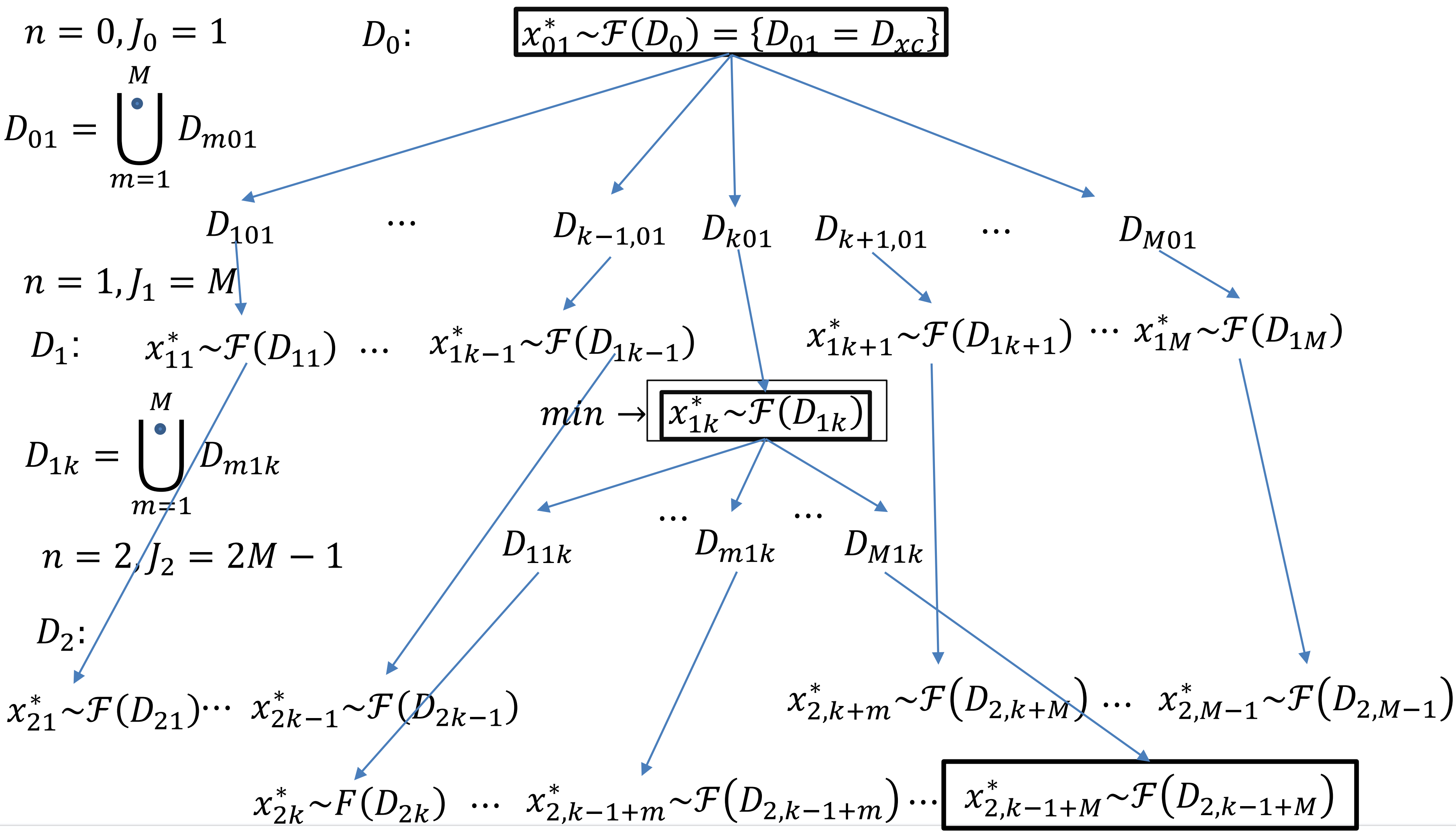
Step 4: $\mathcal{D}^* = \mathcal{D}_{nk}$

$\mathcal{D}_{n+1} = \mathcal{D}_n \cup \{\mathcal{D}_{1nk}, \mathcal{D}_{2nk}, \dots, \mathcal{D}_{Mnk}\} \backslash \mathcal{D}_{nk}$

$\mathcal{D}_{n+1,j} = \mathcal{D}_{nj} \qquad j = 1, 2, \dots, k-1$

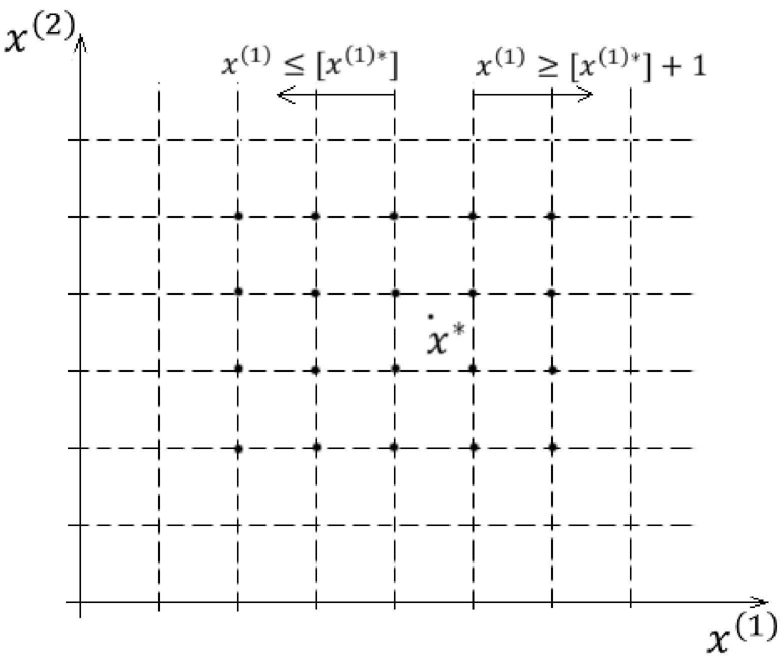
$\mathcal{D}_{n+1,j} = \mathcal{D}_{mnk} \qquad j = k+m, m = 1, 2, \dots, M$

$\mathcal{D}_{n+1,j} = \mathcal{D}_{ni} \qquad j = k+M+i, i = k+1, \dots, J_n, J_{n+1} = J_n + M - 1$


$n = 0, J_0 = 1$
D_0:
$x_{01}^* \sim \mathcal{F}(D_0) = \{D_{01} = D_{xc}\}$
$D_{01} = \dot{\bigcup}_{m=1}^{M} D_{m01}$
D_{101} … $D_{k-1,01}$ D_{k01} $D_{k+1,01}$ … D_{M01}
$n = 1, J_1 = M$
D_1:
$x_{11}^* \sim \mathcal{F}(D_{11})$ … $x_{1k-1}^* \sim \mathcal{F}(D_{1k-1})$
$x_{1k+1}^* \sim \mathcal{F}(D_{1k+1})$ ⋯ $x_{1M}^* \sim \mathcal{F}(D_{1M})$
$min \rightarrow$ $x_{1k}^* \sim \mathcal{F}(D_{1k})$
$D_{1k} = \dot{\bigcup}_{m=1}^{M} D_{m1k}$
D_{11k} … D_{m1k} … D_{M1k}
$n = 2, J_2 = 2M - 1$
D_2:
$x_{21}^* \sim \mathcal{F}(D_{21})$⋯ $x_{2k-1}^* \sim \mathcal{F}(D_{2k-1})$
$x_{2,k+m}^* \sim \mathcal{F}(D_{2,k+M})$ ⋯ $x_{2,M-1}^* \sim \mathcal{F}(D_{2,M-1})$
$x_{2k}^* \sim F(D_{2k})$ … $x_{2,k-1+m}^* \sim \mathcal{F}(D_{2,k-1+m})$ ⋯ $x_{2,k-1+M}^* \sim \mathcal{F}(D_{2,k-1+M})$

$$\mathcal{F}(\mathcal{D}_{xc}) = \min_{x \in \mathcal{D}_x} F(x)$$

Division of the set of feasible solutions



$$x^{(s)} \leq \left[x^{(s)*}\right]$$

$$x^{(s)} \geq \left[x^{(s)*}\right] + 1$$

$$\mathcal{D}_{11} = \mathcal{D}_{01} \cap \{x^{(1)} \leq [x^{(1)*}]\}$$
$$\mathcal{D}_{12} = \mathcal{D}_{01} \cap \{x^{(1)} \geq [x^{(1)*}] + 1\}$$

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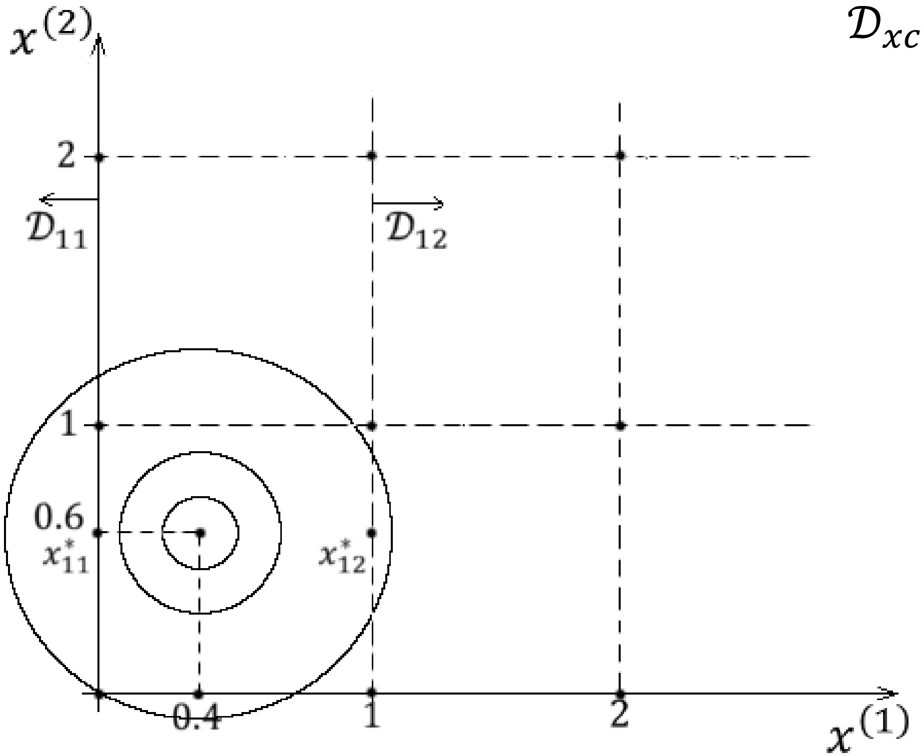
$$F(x) = \left(x^{(1)} - 0.4\right)^2 + \left(x^{(2)} - 0.6\right)^2$$

$$\mathcal{D}_{xc} = \{x \in R^2, x^{(1)}, x^{(2)} \in \mathcal{C}\}$$



1. $\mathcal{F}(\mathcal{D}_{xc}) = \min_{R^2} F(x)$

$x_{01}^{*} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \qquad \mathcal{F}(\mathcal{D}_{01}) = 0$

2. $\mathcal{D}_{11} = \{x \in R^2, x^{(1)}, x^{(2)} \in c \wedge x^{(1)} \leq [0.4] = 0\}$

$\mathcal{D}_{12} = \{x \in R^2, x^{(1)}, x^{(2)} \in c \wedge x^{(1)} \geq [0.4] + 1 = 1\}$

$\mathcal{F}(\mathcal{D}_{11}) = \min_{x^{(1)} \leq 0} F(x)$

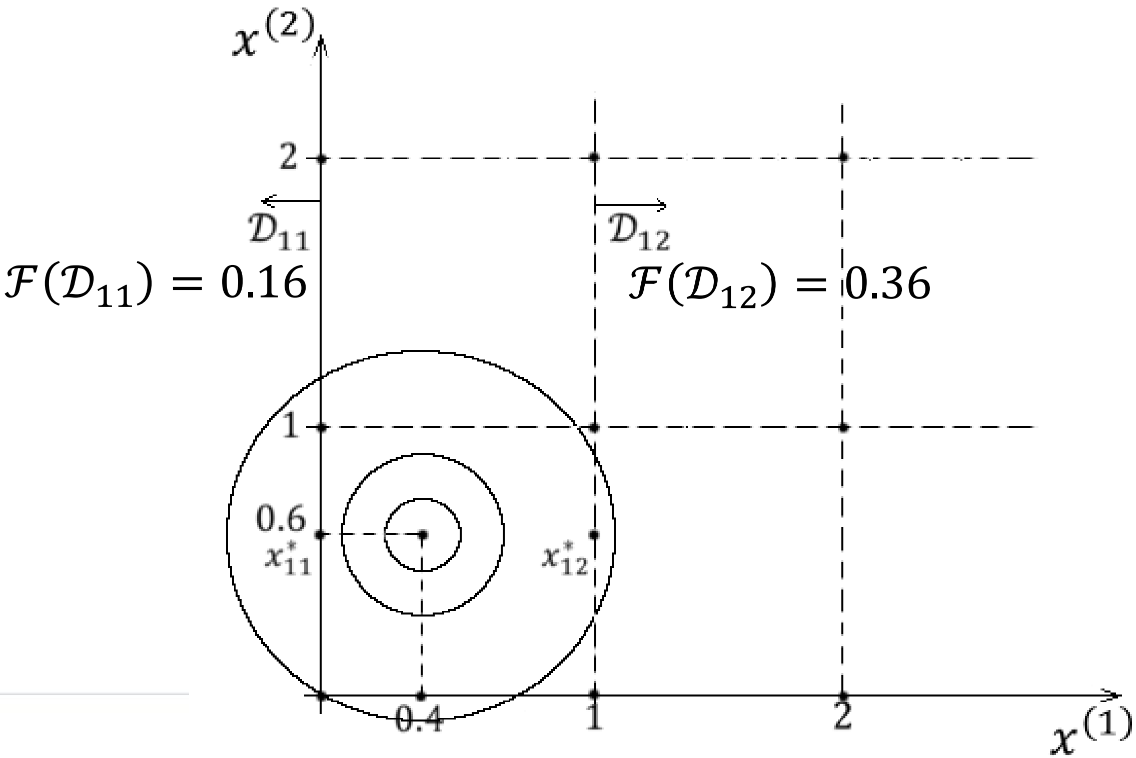
$x_{11}^{*} = \begin{bmatrix} 0 \\ 0.6 \end{bmatrix} \qquad \mathcal{F}(\mathcal{D}_{11}) = (0.4)^2 + 0^2 = 0.16$

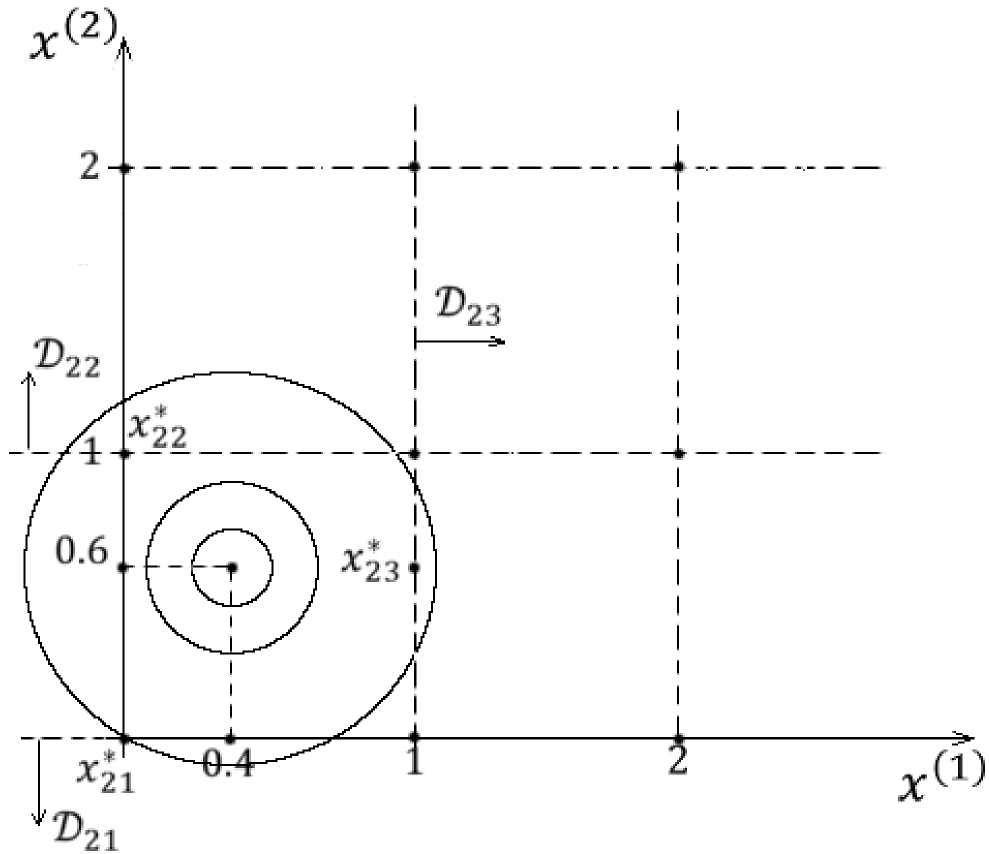
$\mathcal{F}(\mathcal{D}_{12}) = \min_{x^{(1)} \geq 1} F(x)$

$x_{12}^{*} = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix} \qquad \mathcal{F}(\mathcal{D}_{12}) = (0.6)^2 + 0^2 = 0.36$

$$F(x) = \left(x^{(1)} - 0.4\right)^2 + \left(x^{(2)} - 0.6\right)^2$$

$$\mathcal{D}_{xc} = \{x \in R^2, x^{(1)}, x^{(2)} \in \mathcal{C}\}$$




$x^{(2)}$
2
$\mathcal{D}_{23}$
$\mathcal{D}_{22}$
x^*_{22}
1
0.6
x^*_{23}
x^*_{21}
0.4
1
2
$x^{(1)}$
$\mathcal{D}_{21}$

3. $n = 2$

$\mathcal{D}_{21} = \{x \in R^2, x^{(1)}, x^{(2)} \in c \wedge x^{(1)} \leq 0, x^{(2)} \leq [0.6] = 0\}$

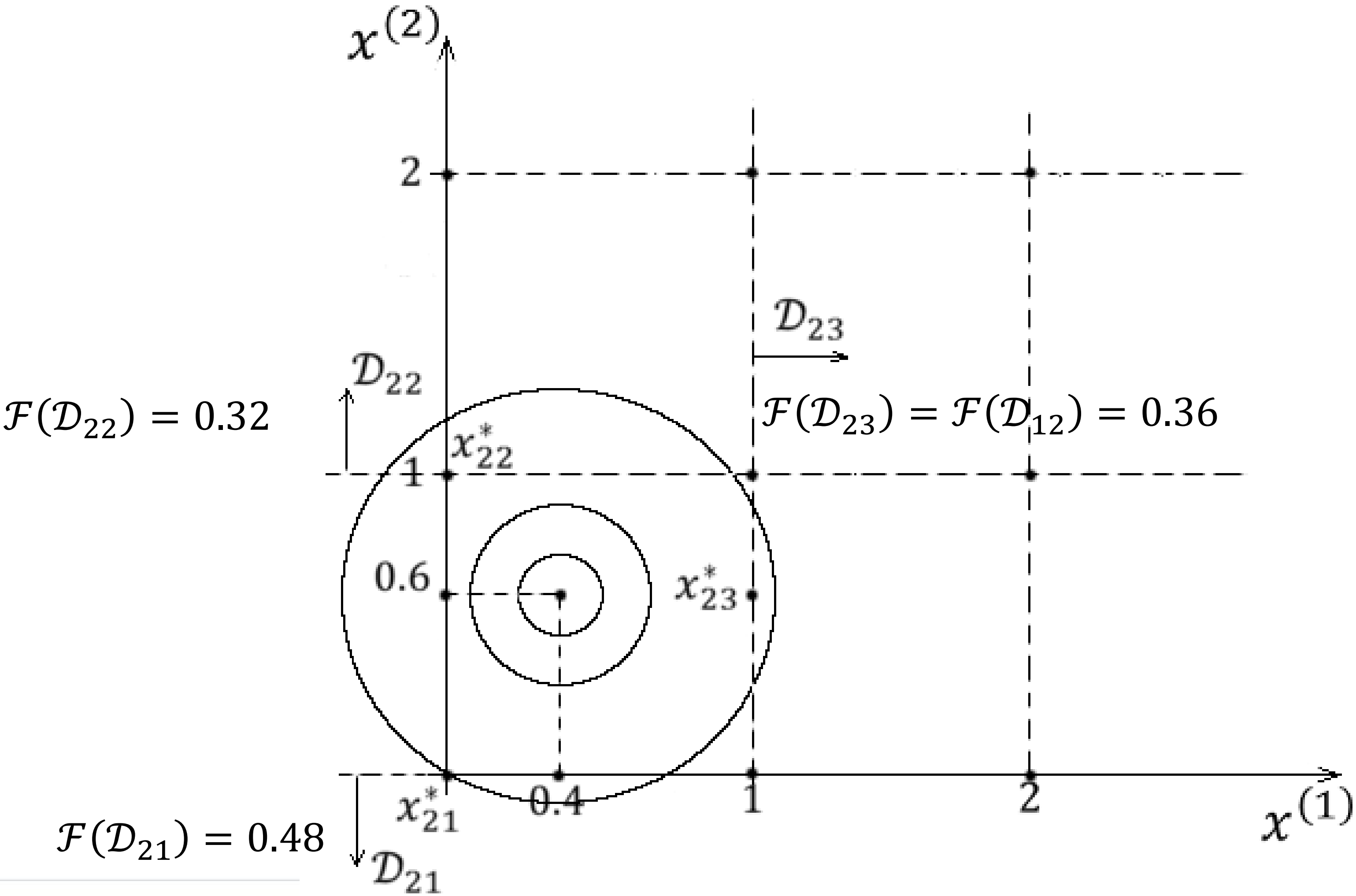
$\mathcal{F}(\mathcal{D}_{21}) = \min_{\substack{x^{(1)} \leq 0 \\ x^{(2)} \leq 0}} F(x) \quad x_{21}^{*} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mathcal{F}(\mathcal{D}_{21}) = 0.16 + 0.32 = 0.48$

$\mathcal{D}_{22} = \{x \in R^2, x^{(1)}, x^{(2)} \in c \quad x^{(1)} \leq 0 \quad x^{(2)} \geq [0.6] + 1 = 1\}$

$\mathcal{F}(\mathcal{D}_{22}) = \min_{\substack{x^{(1)} \leq 0 \\ x^{(2)} \geq 1}} F(x) \quad x_{22}^{*} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathcal{F}(\mathcal{D}_{22}) = 0.16 + 0.16 = 0.32$

$\mathcal{D}_{23} = \mathcal{D}_{12}$

$\mathcal{F}(\mathcal{D}_{23}) = 0.36$


$x^{(2)}$
2
$\mathcal{D}_{23}$
$\mathcal{D}_{22}$
$\mathcal{F}(\mathcal{D}_{22}) = 0.32$
x_{22}^{*}
$\mathcal{F}(\mathcal{D}_{23}) = \mathcal{F}(\mathcal{D}_{12}) = 0.36$
1
0.6
x_{23}^{*}
x_{21}^{*}
0.4
1
2
$x^{(1)}$
$\mathcal{F}(\mathcal{D}_{21}) = 0.48$
$\mathcal{D}_{21}$

$n = 0, J_0 = 1$ D_0: $x_{01}^* = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \sim \mathcal{F}(D_{01}) = \min_{R^2} F(x) = 0$

$x^{(1)} \leq [0.4] = 0$ $x^{(1)} \geq [0.4] + 1 = 1$

$n = 1, J_1 = 2$

D_1: $x_{11}^* = \begin{bmatrix} 0 \\ 0.6 \end{bmatrix} \sim \mathcal{F}(\mathcal{D}_{11}) = \min_{x^{(1)} \leq 0} F(x) = 0.16$ $x_{12}^* = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix} \sim \mathcal{F}(\mathcal{D}_{12}) = \min_{x^{(1)} \geq 1} F(x) = 0.36$

$x^{(1)} \leq 0, x^{(2)} \leq [0.6] = 0$ $x^{(2)} \geq [0.6] + 1 = 1$

$n = 2, J_2 = 3$

D_2: $x_{21}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \sim \mathcal{F}(\mathcal{D}_{21}) = \min_{\substack{x^{(1)} \leq 0 \\ x^{(2)} \leq 0}} F(x) = 0.48$ $x_{22}^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sim \mathcal{F}(\mathcal{D}_{22}) = \min_{\substack{x^{(1)} \leq 0 \\ x^{(2)} \geq 1}} F(x) = 0.32$

$\mathcal{D}_{23} = \mathcal{D}_{12}$

$x_{12}^* = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix} \sim \mathcal{F}(\mathcal{D}_{23}) = 0.36$

Thank you for attention