Computer Science

Jerzy Świątek

Systems Modelling and Analysis

Choose yourself and new technologies

L.17.d Constrained optimization methods

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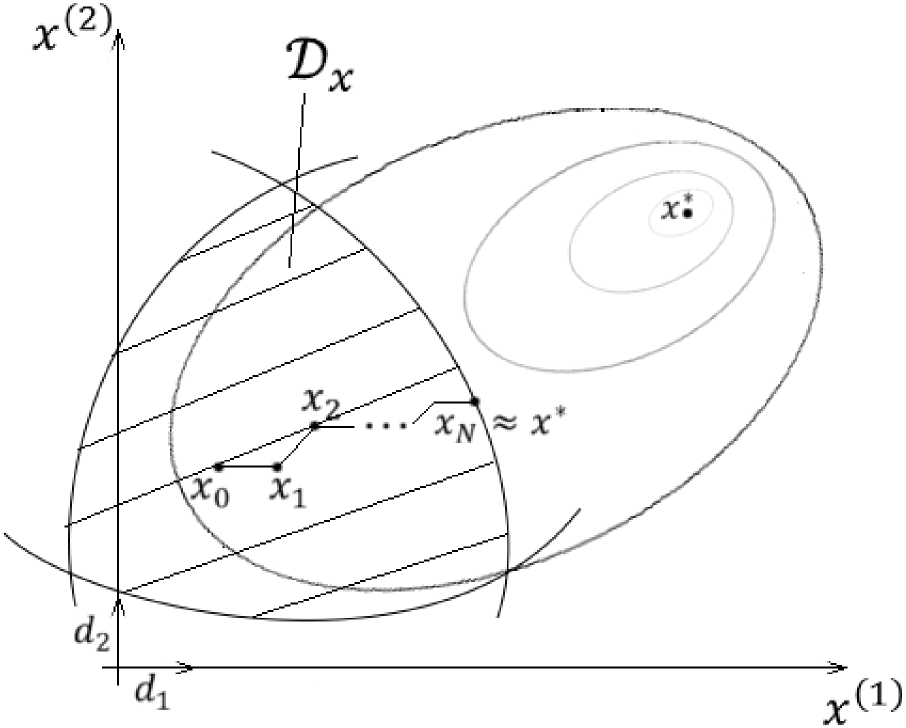
Numerical constrained optimization methods



$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

1. Elimination of constraints
2. Penalty function method
   - exterior penalty
   - barrier function
3. Methods of feasible directions
4. Other approaches

Elimination of constraints

$$x^* \to F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

$$x^* \longrightarrow \min_{x \in D_x} F(x)$$

$$\uparrow \qquad\qquad \downarrow$$

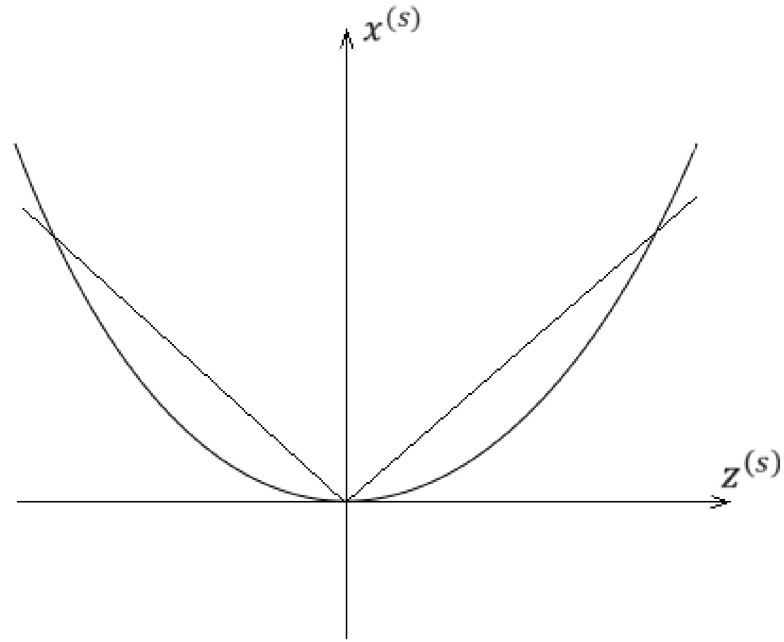
$$p: R^s \to \mathcal{D}_x$$
$$x = p(z)$$
$$\bar{F}(z) = F\big(p(z)\big)$$
$$z \in R^s \to x \in \mathcal{D}_x$$

$$z^* \longleftarrow \min_{z \in R^s} \bar{F}(z)$$

Example

$$x^* \to F(x^*) = \min_{x \in \mathcal{D}_x} F(x) \qquad F(x) = \sum_{s=1}^{S} \left(x^{(s)} + 5\right)^2$$

$$D_x = \left\{x \in R^S, \quad x^{(s)} \geq 0, \quad s = 1, 2, \dots, S\right\}$$



$$x^{(s)} = \left(z^{(s)}\right)^2 \text{ or } x^{(s)} = \left|z^{(s)}\right|$$

$$z^{(s)} \in R \quad \to x^{(s)} \in [0, \infty)$$

$$F(x) = \sum_{s=1}^{S} \left(x^{(s)} + 5\right)^2 \quad \to \quad \bar{F}(z) = \sum_{s=1}^{S} \left(\left|z^{(s)}\right| + 5\right)^2 \qquad z^* \to \bar{F}(z^*) = \min_{z \epsilon R^S} \bar{F}(z)$$

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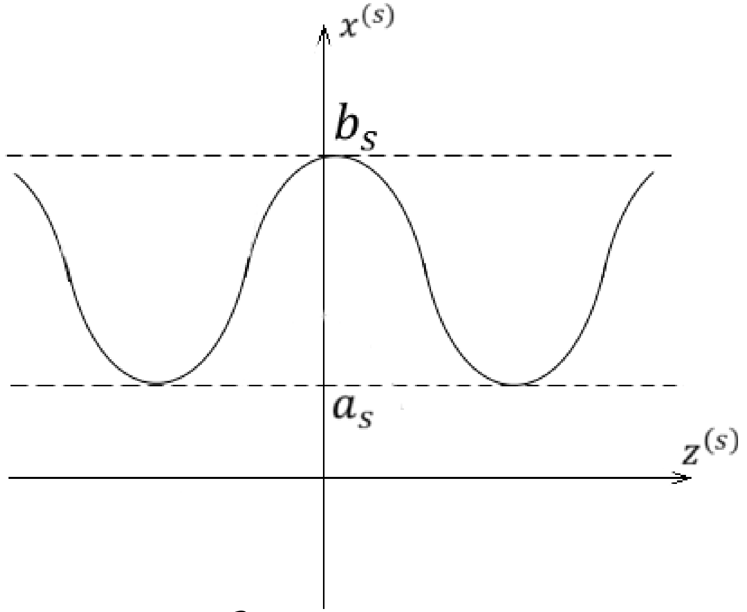
Example

$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_x} F(x) \qquad F(x) = \sum_{s=1}^{S} \left(x^{(s)}\right)^2$$

$$\mathcal{D}_x = \left\{x \in R^S, \quad a_s \leq x^{(s)} \leq b_s, \quad s = 1, 2, \dots, S\right\}$$

$$x^{(s)} = a_s + (b_s - a_s)\sin^2 z^{(s)}$$

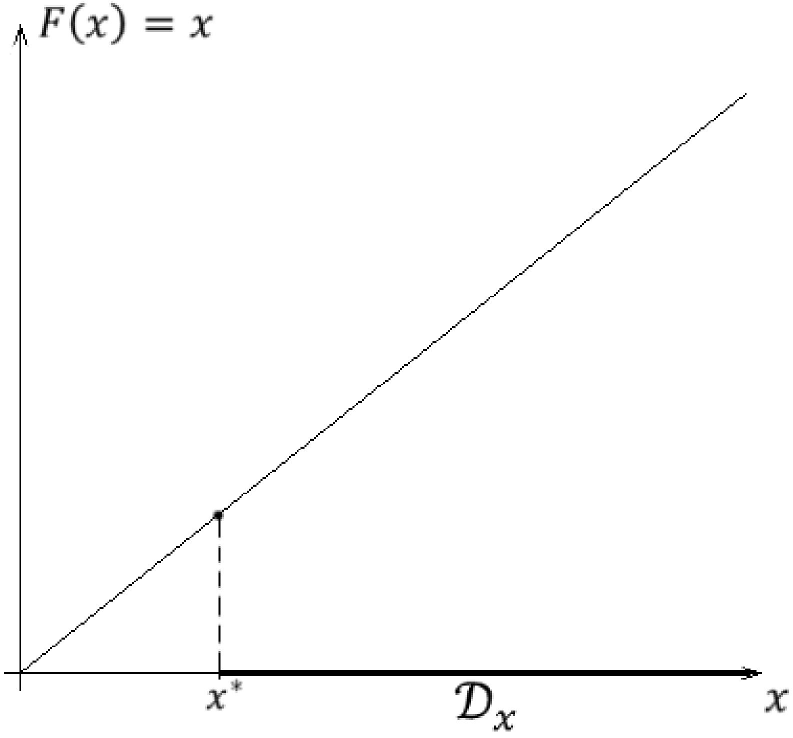
$$z^{(s)} \in R^1 \rightarrow x^{(2)} \in [a_s, b_s]$$



$$F(x) = \sum_{s=1}^{S} \left(x^{(s)}\right)^2 \rightarrow \bar{F}(z) = \sum_{s=1}^{S} \left(a_s + (b_s - a_s)\sin^2 z^{(s)}\right)^2 \quad z^* \rightarrow \bar{F}(z^*) = \min_{z \epsilon R^S} \bar{F}(z)$$

Example

$$x^* \to F(x^*) = \min_{x \in \mathcal{D}_x} F(x) \qquad F(x) = x \quad \mathcal{D}_x = \{x \in R^1 \ \ x \geq 1\}$$



$$x^* \to \min_{x \geq 1} x$$

$$x = z^2 + 1 = p(z)$$

$$z \in R \to x \in [1, \infty)$$

$$\bar{F}(z) = z^2 + 1$$

$$z^* \to \min_{z \epsilon R} \bar{F}(z)$$

$$z^* \to \min_{z \epsilon R} (z^2 + 1)$$

$$\left(z^2 + 1\right)' = 2z = 0$$

$$z^* = 0$$

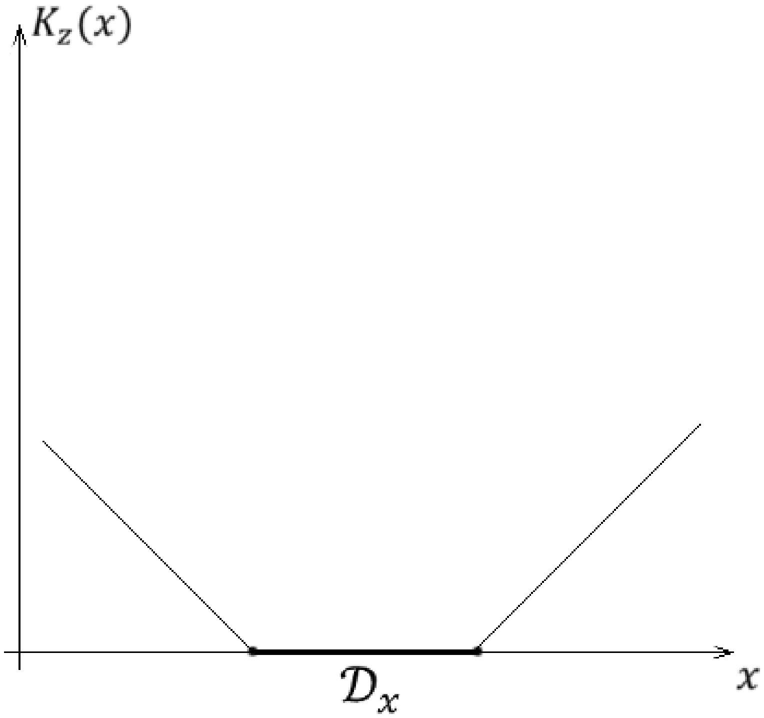
$$x^* = p(z^*) = (z^*)^2 + 1 = 1$$

Penalty function method
Exterior penalty function

$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

$$F_k(x) = F(x) + r_k K_z(x)$$

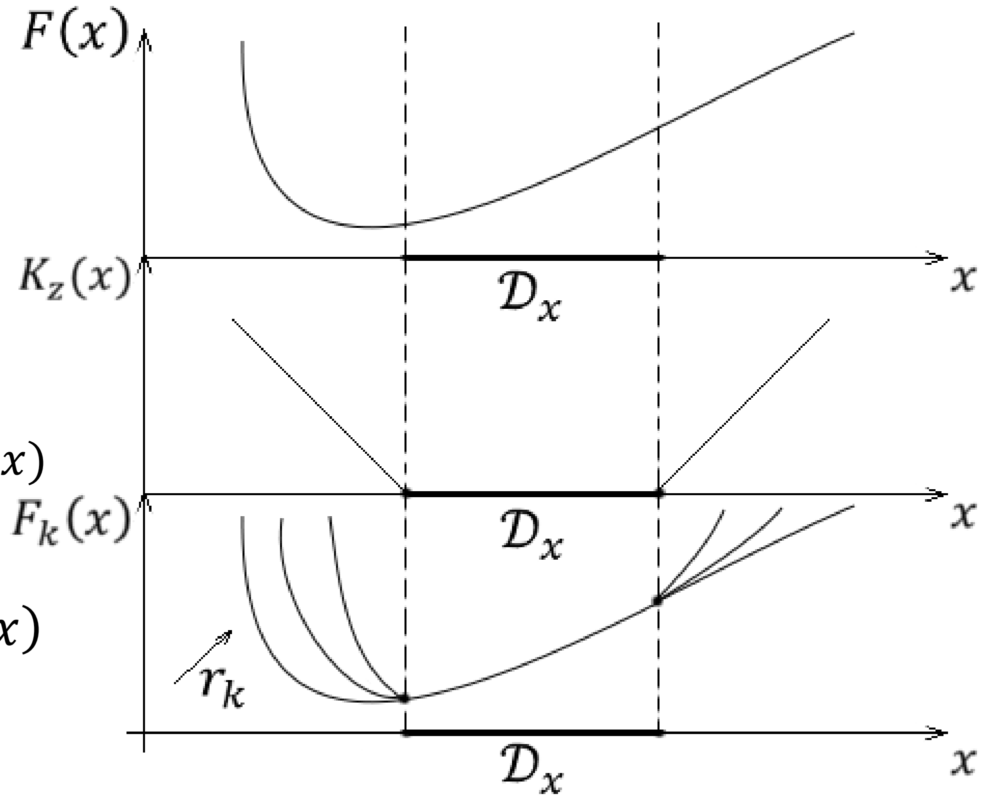
$$K_z(x) \begin{cases} = 0 & x \in \mathcal{D}_x \\ > 0 & x \notin \mathcal{D}_x \end{cases}$$



$$x^* \to F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

$$F_k(x) = F(x) + r_k K_w(x)$$

$$x_k^* \to F\left(x_k^*\right) = \min_{x \in R^S} F_k(x)$$



$r_k > 0$ $\lim_{k \to \infty} r_k = \infty$

Example of exterior penalty function

$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

$$\mathcal{D}_x = \{x \in R^s,\ \varphi_l(x) = 0, \quad l = 1, 2, \dots, L, \quad \psi_m(x) \le 0, \quad m = 1, 2, \dots, M\}$$

$$\varphi_l(x) \rightarrow K_{lz}(x) = \left(\varphi_l(x)\right)^2, l = 1, 2, \dots, L$$

$$\psi_m(x) \rightarrow K_{mz}(x) = (\max\{0, \psi_m(x)\}) \quad , m = 1, 2, \dots, M$$

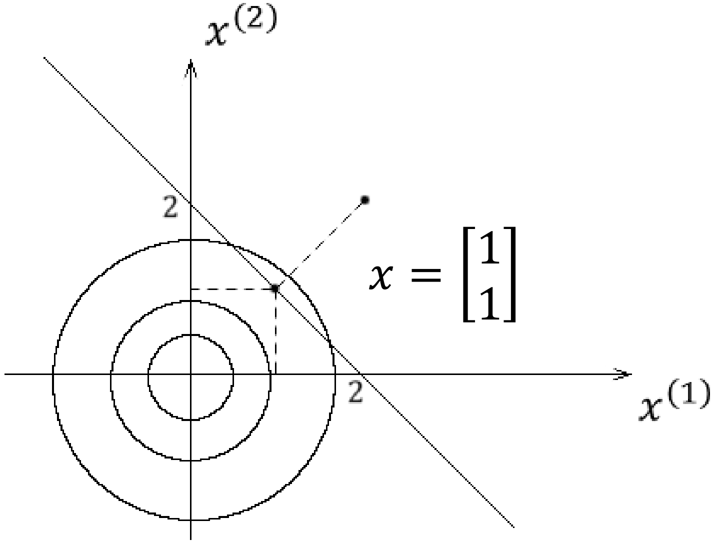
$$K_w(x) = \sum_{l=1}^{L} r_l\left(\varphi_l(x)\right)^2 + \sum_{m=1}^{M} \rho_m \ \max\{0, \psi_m(x)\}$$

Example

$$x^* \to F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

$$F(x) = \left(x^{(1)}\right)^2 + \left(x^{(2)}\right)^2$$

$$\mathcal{D}_x = \{x \in R^2, x^{(1)} + x^{(2)} - 2 = 0\}$$



$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$F_k(x) = \left(x^{(1)}\right)^2 + \left(x^{(2)}\right)^2 + r_k\left(x^{(1)} + x^{(2)} - 2\right)^2$$

$$F_k'(x) = 0$$

$$2x^{(1)} + 2r_k\left(x^{(1)} + x^{(2)} - 2\right) = 0$$

$$2x^{(2)} + 2r_k\left(x^{(1)} + x^{(2)} - 2\right) = 0$$

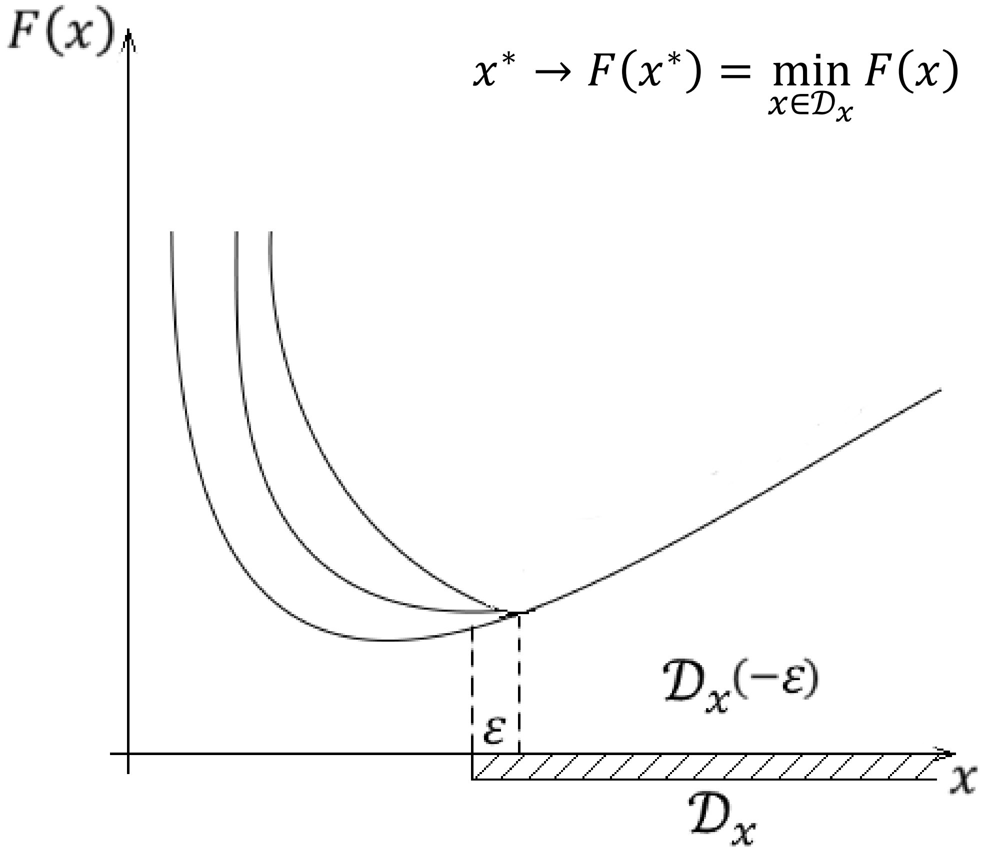
$$x^{(1)} = x^{(2)}$$

$$+2r_k\left(x^{(1)} + x^{(2)} - 2\right) = 0$$

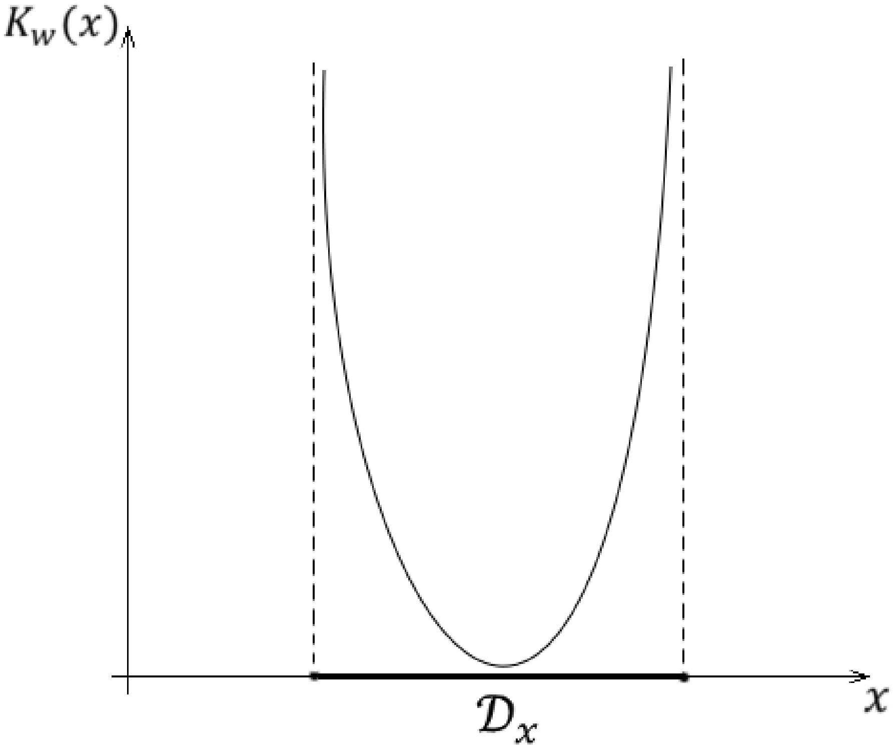
$$x^{(1)} = x^{(2)} = \frac{2r_k}{2r_k + 1}$$

$$x^{(1)} = x^{(2)} = \lim_{r_k \to \infty} \frac{2r_k}{2r_k + 1} = 1, \qquad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Modifications



Barrier function



$$x^* \to F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

$$F_k(x) = F(x) + r_k K_w(x)$$

$K_w(x)$ – such a function, that

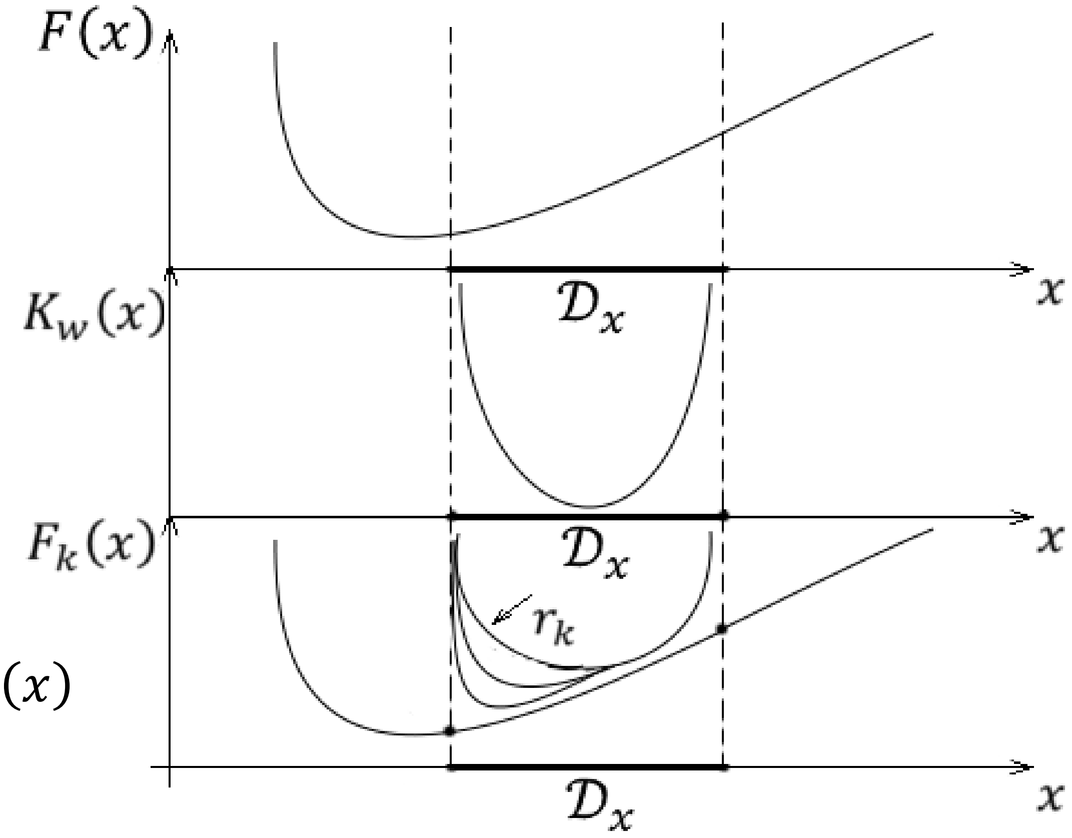
$$\exists\ x_1, x_{,2}, \ldots, x_n \in \mathcal{D}_x \qquad \lim_{k \to \infty} x_k = x \in \mathcal{D}_x$$

$$\exists_k \quad K_w(x_k + 1) > K_w(x_k)$$

$$x^* \to F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

$$F_k(x) = F(x) + r_k K_w(x)$$

$$x_k^* \to F\left(x_k^*\right) = \min_{x \in R^S} F_k(x)$$



$$r_k > 0 \qquad \lim_{k \to \infty} r_k = 0$$

$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

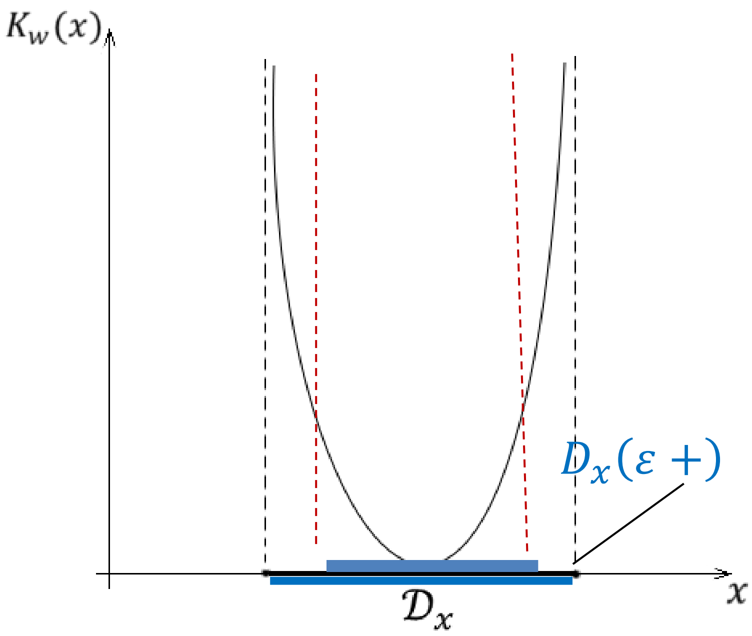
$$\mathcal{D}_x = \{x \in R^s, \qquad \psi_m(x) \leq 0, \qquad m = 1, 2, \ldots, M\}$$

$$K_{Wm}(x) = \frac{-1}{\psi_m(x)}$$

$$K_W(x) = \sum_{m=1}^{M} r_m K_{Wm}(x) = \sum_{m=1}^{M} r_m \frac{-1}{\psi_m(x)}, \quad m = 1, 2, \ldots, M$$

Modifications

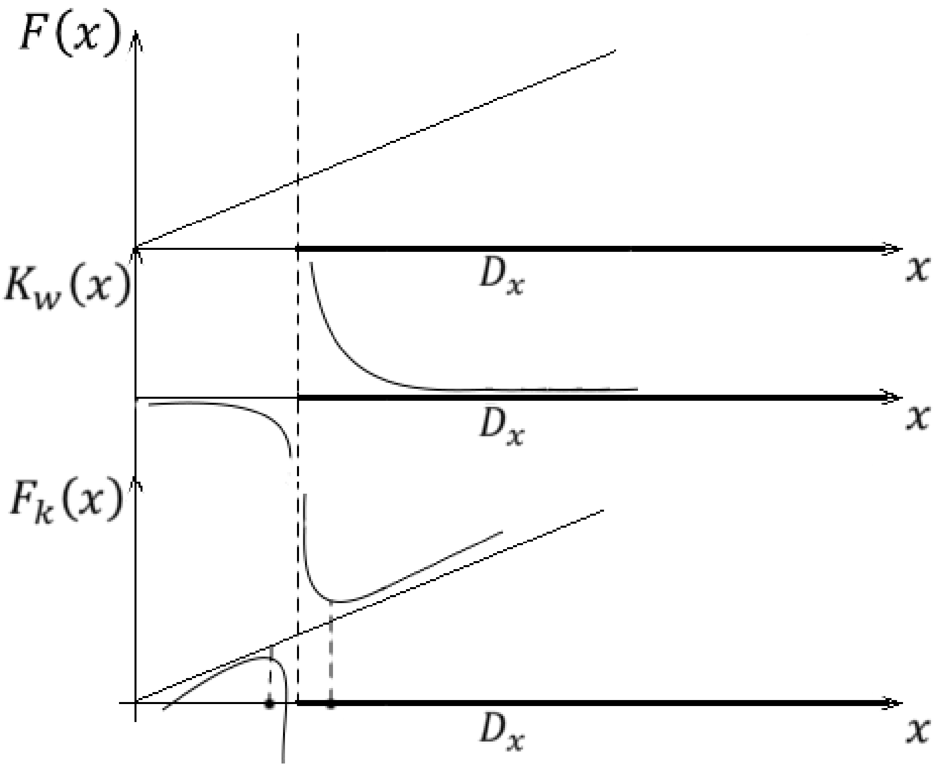
$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$



Example

$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

$$F(x) = x;\ \ \mathcal{D}_x = \{x \in R^1, x \geq 1\} \equiv \{x \in R^1, 1 - x \leq 0\}$$



$$K_w(x) = \frac{-1}{1-x} = \frac{1}{x-1}$$

$$F_k(x) = x + r_k \frac{1}{x-1}$$

$$F'(x) = 1 + \frac{-r_k}{(x-1)^2} = 0$$

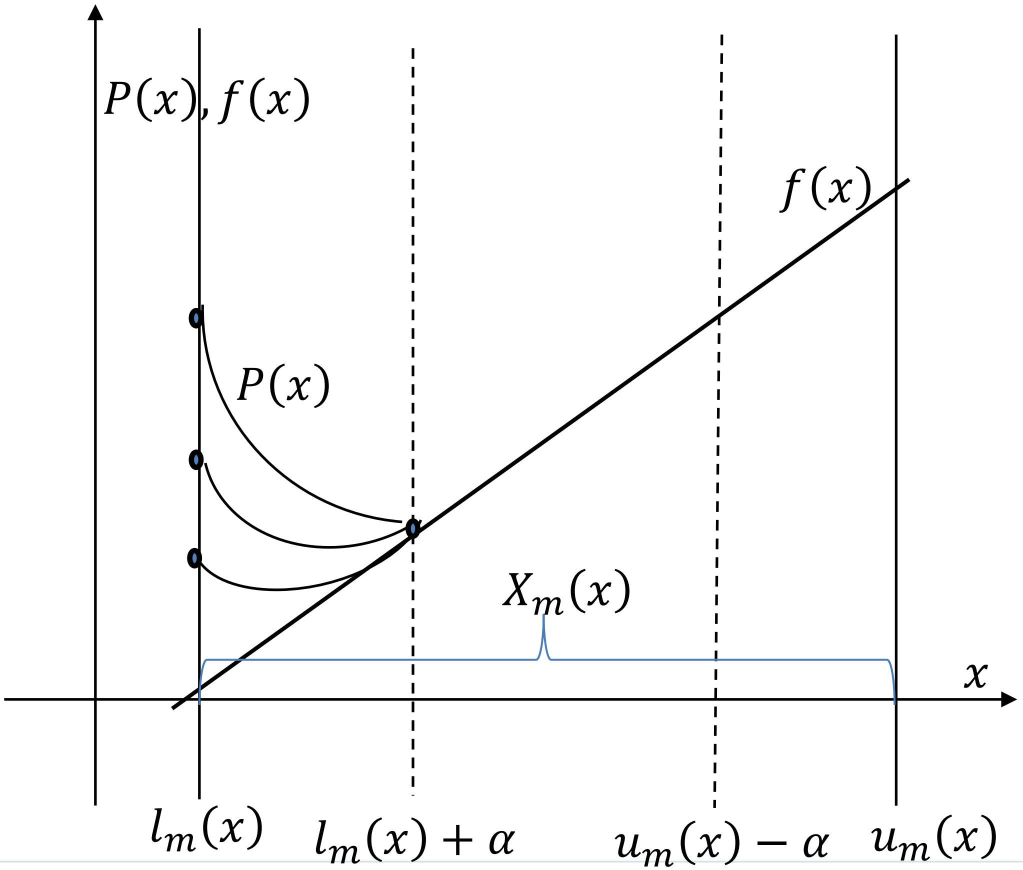
$$x_k = 1 \pm \sqrt{r_k}$$

$$x_k = 1 - \sqrt{r_k} \notin \mathcal{D}_x$$

$$x_k = 1 + \sqrt{r_k} \in \mathcal{D}_x$$

$$\lim_{n \to \infty} x_n = 1$$

Rozebrock Method



Rozebrocka Method

$$x^* \to F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

$$D_x = \{l_m(x) \leq X_m(x) \leq u_m(x) \;\; m = 1,2,\cdots,M\}$$

$$l_m(x) \leq X_m(x) \leq l_m(x) + \alpha$$

$$u_m(x) \leq X_m(x) \leq u_m(x) - \alpha$$

$$P(x) = f(x) - (f(x) - f^*)\left(3\mu - 4\mu^2 + 2\mu^3\right)$$

$$\mu = \frac{l_m(x) + \alpha - X_m(x)}{\alpha} \text{ or } \mu = \frac{X_m(x) - u_m(x) + \alpha}{\alpha}$$

$$l_m(x) + \alpha \leq X_m(x) \leq u_m(x) - \alpha$$

$$P(x) = f(x)$$

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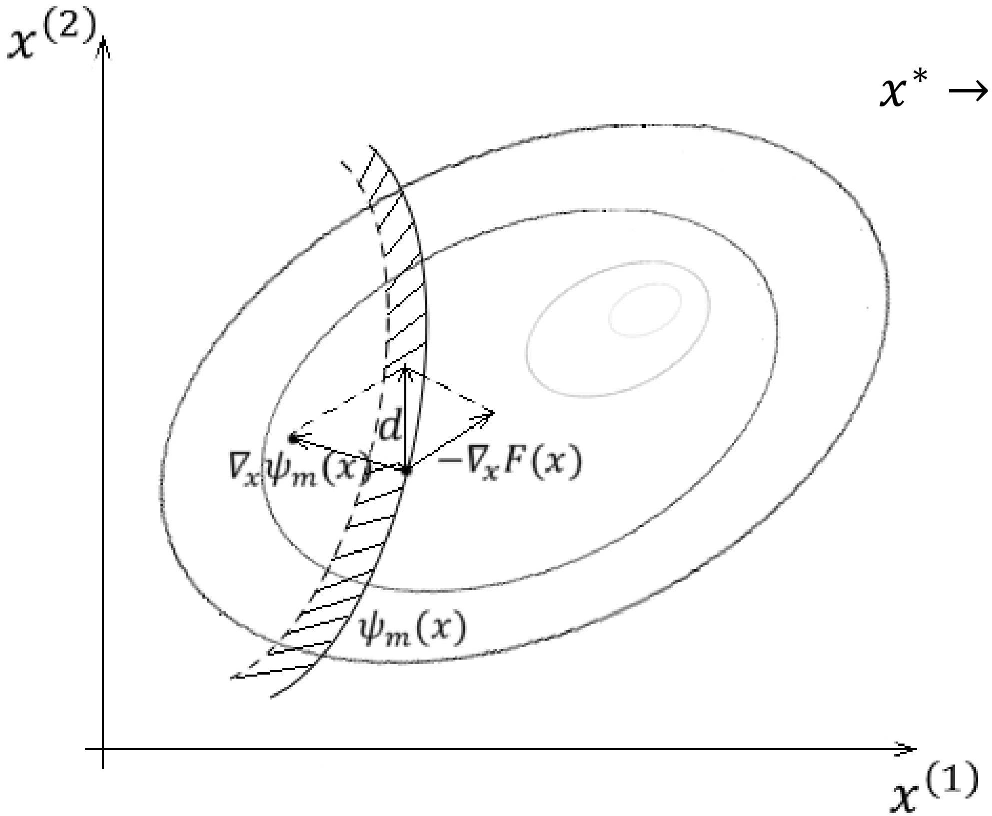
Feasible directions method

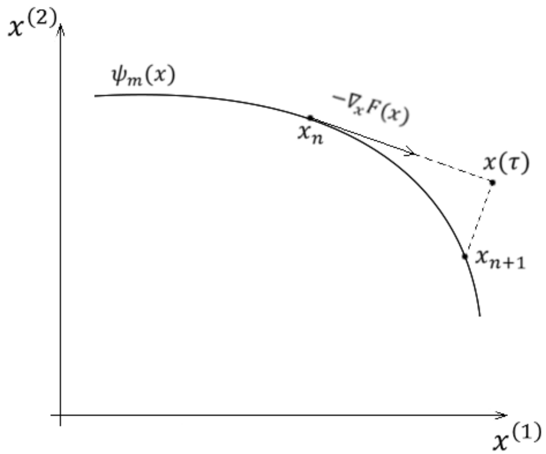


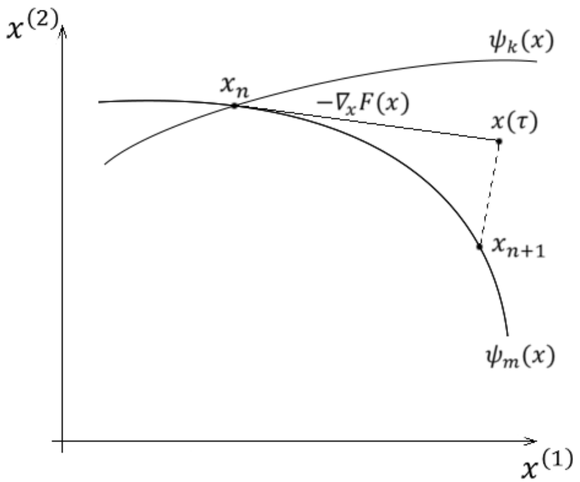
$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

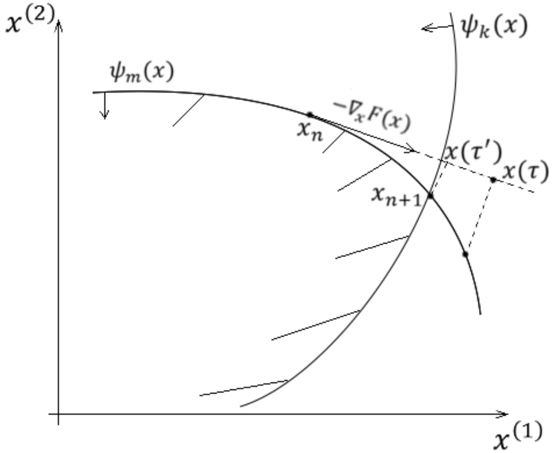
$$d = \frac{\nabla_x \psi_m(x)}{\|\nabla_x \psi_m(x)\|} - \frac{\nabla_x F(x)}{\|\nabla_x F(x)\|}$$

$$x: \psi(x) - \delta \leq 0$$

Gradient projection method of Rosen

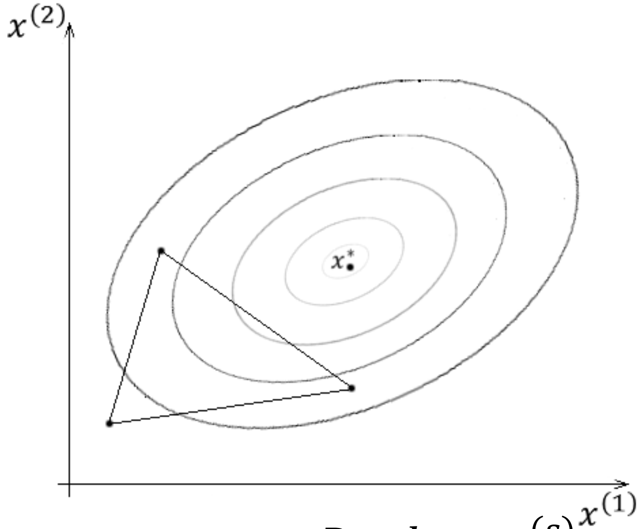






Complex Method (Box) Nelder-Mead Method (simplex)

$x_1\ x_2\ \dots x_K$ - complex $in\ S$ - domine space $K \geq S+1$



$$x_H \rightarrow F(x_H) = \max_{1 \leq s \leq K} F(x_s)$$

$$\bar{x} = \frac{1}{S} \sum_{s=1, s \neq H}^{K} x_s$$

$$x^* = (1+\alpha)\bar{x} \text{-} x_H$$

$$D_x\colon\ l_s \leq x^{(s)} \leq u_s\ s = 1, 2, \dots, S$$
$$l_m \leq x_m(x) \leq u_m\ m = 1, 2, \dots, M$$

x_k=$l_k + r_k|u_k - l_k|, r_k\ - random\ number \in [0,1], k = 1, 2, \dots, K$

$If\ x_k\ is\ not\ element\ of\ D_x\ then\ move\ point\ in\ the\ direction\ of\ centroid\ of\ accepted\ points$

Complex Method

Data: $x_0, c, \varepsilon, \mathrm{K}, \alpha$ $recomended\ K = 2S, \alpha = 1.3$
Step 0: $x_1\ x_2$ ... $x_K \in D_x$ - initial complex, $n = 0$
Step 1: Determine the radius of the minimum ball containing the complex - ρ_{min}
Step 2: Reviev if $\rho_{min} < \varepsilon$ If yes then stop, otherwise go to step 3
Step 3: $x_H \rightarrow F(x_H) = \max\limits_{1 \le s \le S+1} F(x_s)$,

$\bar{x} = \frac{1}{S} \sum_{\substack{s=1 \\ s \ne H}}^{K} x_s$ reviev if $\bar{x} \in D_x$ if not add point to the complex

Step 4: $x^* = (1 + \alpha)\bar{x} + \alpha x_H$
Step 5: Reviev if $x^* \in D_x$ if yes then go to step 7, otherwise
Step 6: $x^* = \bar{x} - (\bar{x} - x^*)/2$ aż $x^* \in D_x$
Step 7: If $F(x^*) < F(x_H)$ go to step 2
otherwise go to step 6

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Random search - Down Hill method

Data: $F(x), x_0, D_x, N$

Step 0: n=0, $x^* = x_n$

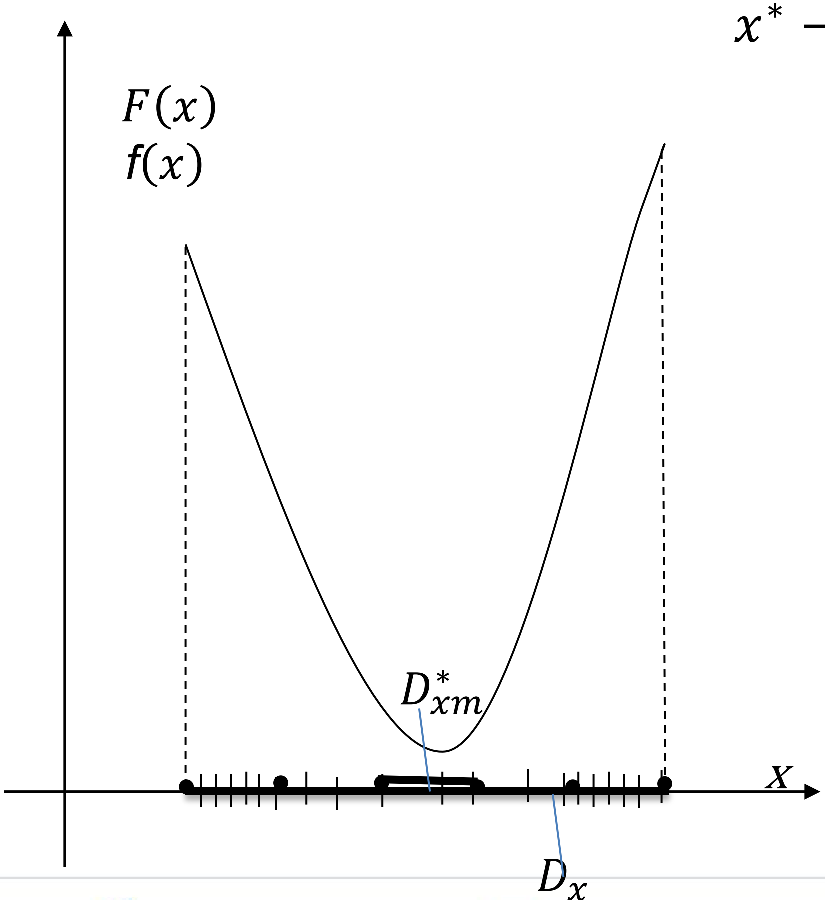
Step 1: Generate point x_{n+1} in the set D_x with unity probability density

Step 2: IF $F(x_{n+1}) < F(x^*)$ THEN $x^* = x_{n+1}$

Step 3: IF $n < N$ THEN $n = n+1$ GO TO STEP 1

Step 4: $x^* = x_N$

Random search



$$x^* \rightarrow F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

$$f(x) = \frac{F(x)}{\int_{D_x} F(x) dx}$$

Random search

Data: $F(x), D_x, \varepsilon, N, M$

Step 1: Generate N point in set D_x with probability density

$$f(x) = \frac{F(x)}{\int_{D_x} F(x)dx}$$

Step 2: Divide set D_x for M disjoint sets such that

$$D_x = \dot{\bigcup_{m=1}^{M}} D_{xm}, \quad \|D_{xm}\| = \frac{1}{M}\|D_x\|$$

Step 3. Count points in all sets D_{xm}

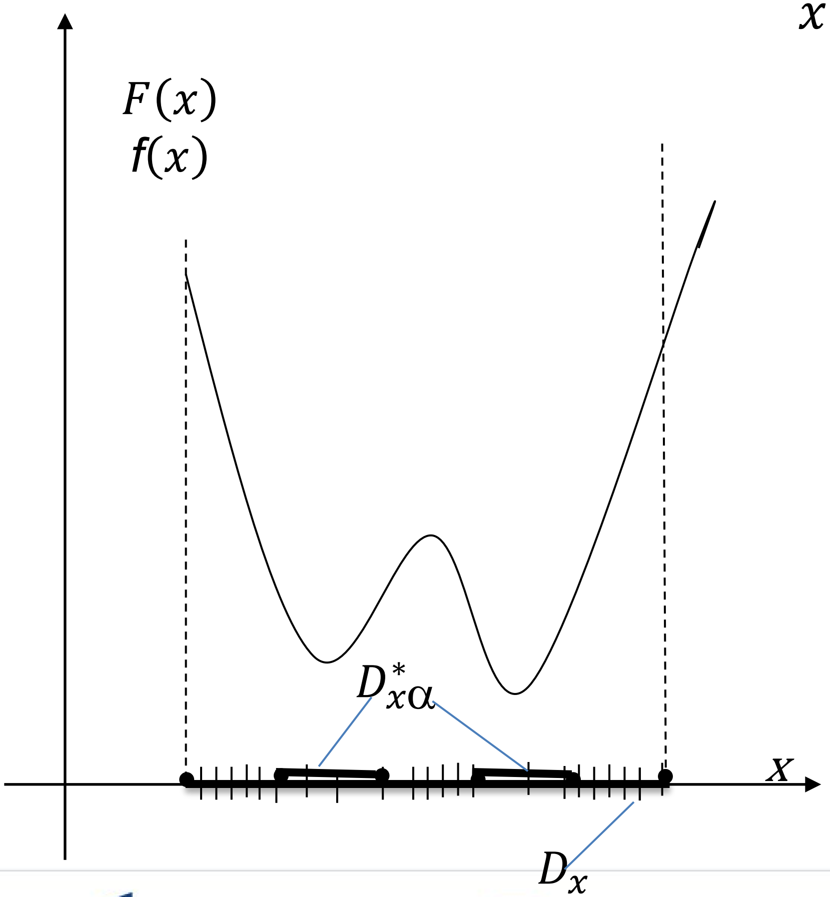
$$N_m - number\ of\ po\text{int}\, s\ in\ the\ set\ D_{xm}$$

Step 4. For the next division choose such set that

$$D^*_{xm} \to N^*_m = \min_{1 \le m \le M}\{N_m\}\ if\ \|D_{xm}\| < \varepsilon\ stop, \forall po\text{int} \in D^*_{xm}\ solution$$

Otrherwise i.e. $\|D^*_{xm}\| \ge \varepsilon\ in\ the\ place\ D_x := D^*_{xm}$ and go to step 1.

Random search



$$x^* \to F(x^*) = \min_{x \in \mathcal{D}_x} F(x)$$

$$f(x) = \frac{F(x)}{\int_{D_x} F(x) dx}$$

Random search

Data: $F(x), D_x, \varepsilon, N, M$

Step 1: Generate N point in set D_x with probability density

$$f(x) = \frac{F(x)}{\int_{D_x} F(x)dx}$$

Step 2: Divide set D_x for M disjoint sets such that

$$D_x = \dot{\bigcup_{m=1}^{M}} D_{xm}, \quad \|D_{xm}\| = \frac{1}{M}\|D_x\|$$

Step 3. Count points in all sets D_{xm}

$$N_m - number\ of\ points\ in\ the\ set\ D_{xm}$$

Step 4. For the next division choose such set that

$$D^*_{xm\alpha} \rightarrow N^*_{m\alpha} \leq \alpha,\ D^*_{x\alpha} = \bigcup_m D^*_{xm\alpha}\ if\ \|D^*_{x\alpha}\| < \varepsilon\ stop,\ \forall point \in D^*_{x\alpha}\ solution$$

Otrherwise i.e. $\|D^*_{xm}\| \geq \varepsilon$ in the place $D_x := D^*_{xm}$ and go to step 1.

Random search – random direction choice

Data: $F(x), x_0, r, N, \varepsilon$

Step 0: $n=0$, $x^* = x_n$

Step 1: Generate N points with unity probability density on the cycle around point x_n

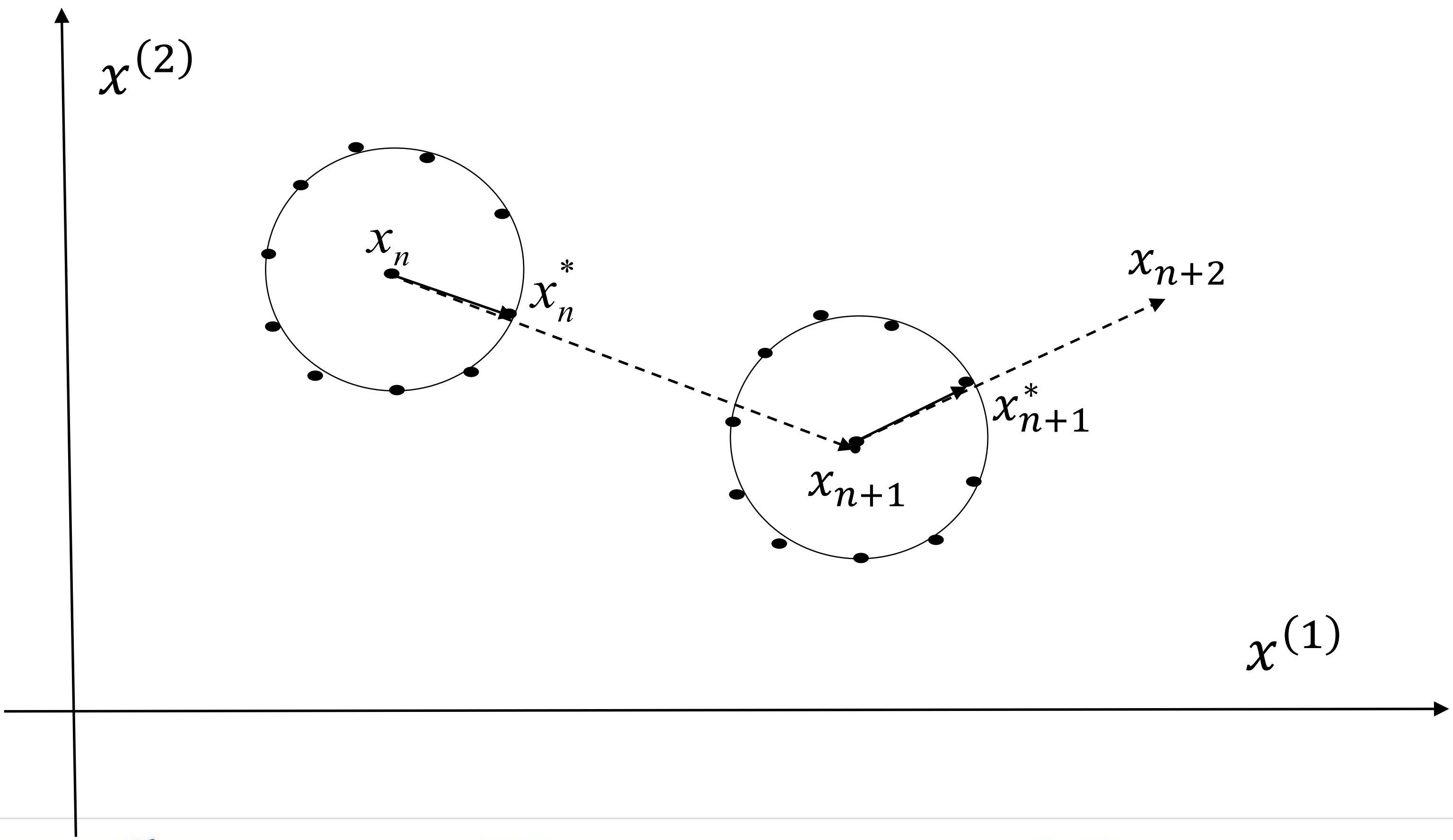
Step 2: Select from generated poits point x_n^* for which thr goal function obtain minimum

Step 3: determine direction $$d = \frac{x_n^* - x_n}{\left\| x_n^* - x_n \right\|}$$

Step 4: Determine point x_{n+1} as minimum in direction d from point x_n

Step 5: IF minimum THEN STOP $x_n \approx x^*$ ELSE

$n=n+1$ GO O STEP 1


$x^{(2)}$
x_n
x_n^*
x_{n+2}
x_{n+1}^*
x_{n+1}
$x^{(1)}$

Nature-Inspired Algorithms
Bibliogrphy

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- Zastosowanie Algorytmów Rojowych do Optymalizacji Parametrów w Modelach Układów Regulacji, Mirosław Tomera, Zeszyty Naukowe Wydziału Elektrotechniki i Automatyki Politechniki Gdańskiej Nr 46, 2015
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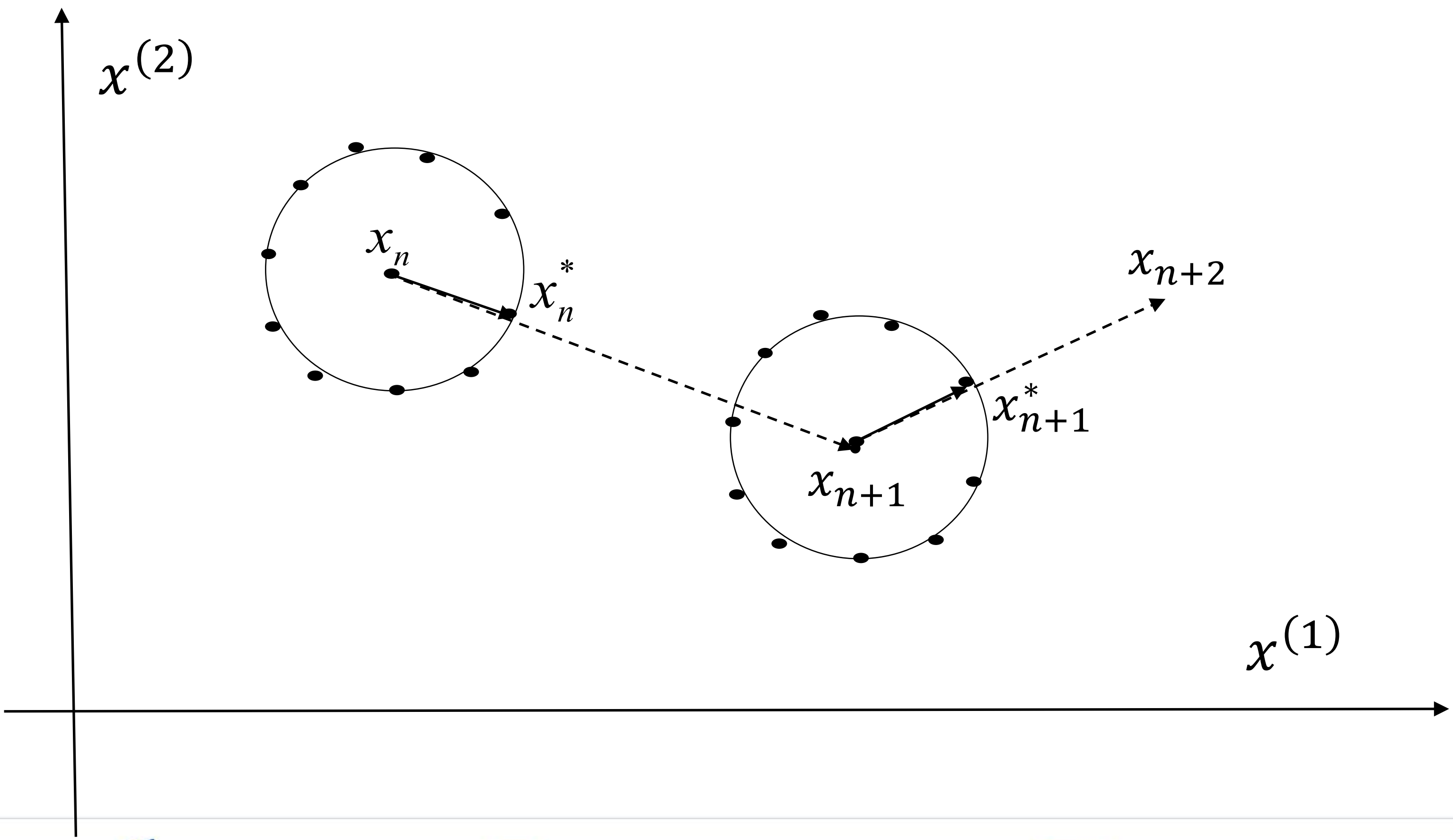
Ant algorithm

- Ants search the space randomly.
- If they come across "food", i.e. a solution – they return to the starting point, leaving a trace of pheromone.
- Encountering a pheromone trace, they follow the designated path, thereby increasing the amount of pheromone.
- The pheromone evaporates over time. Long distances lose the intensity of the pheromone faster.


$x^{(2)}$
x_n
x_n^*
x_{n+2}
x_{n+1}^*
x_{n+1}
$x^{(1)}$

Bee algorithm

We distinguish groups:

- ***Scouts*** – search the solutions randomly
- ***Employees*** – they bring information about the quality of the solutions to which they are currently assigned.
- ***Observers*** – choose the next place based on information from *employees*.

Bee algorithm

- Scouts are sent to random places.
- In the vicinity of the found good values, more individuals are sent to study the local optimum.
- In each iteration, scouts are sent to further explore the solution space.

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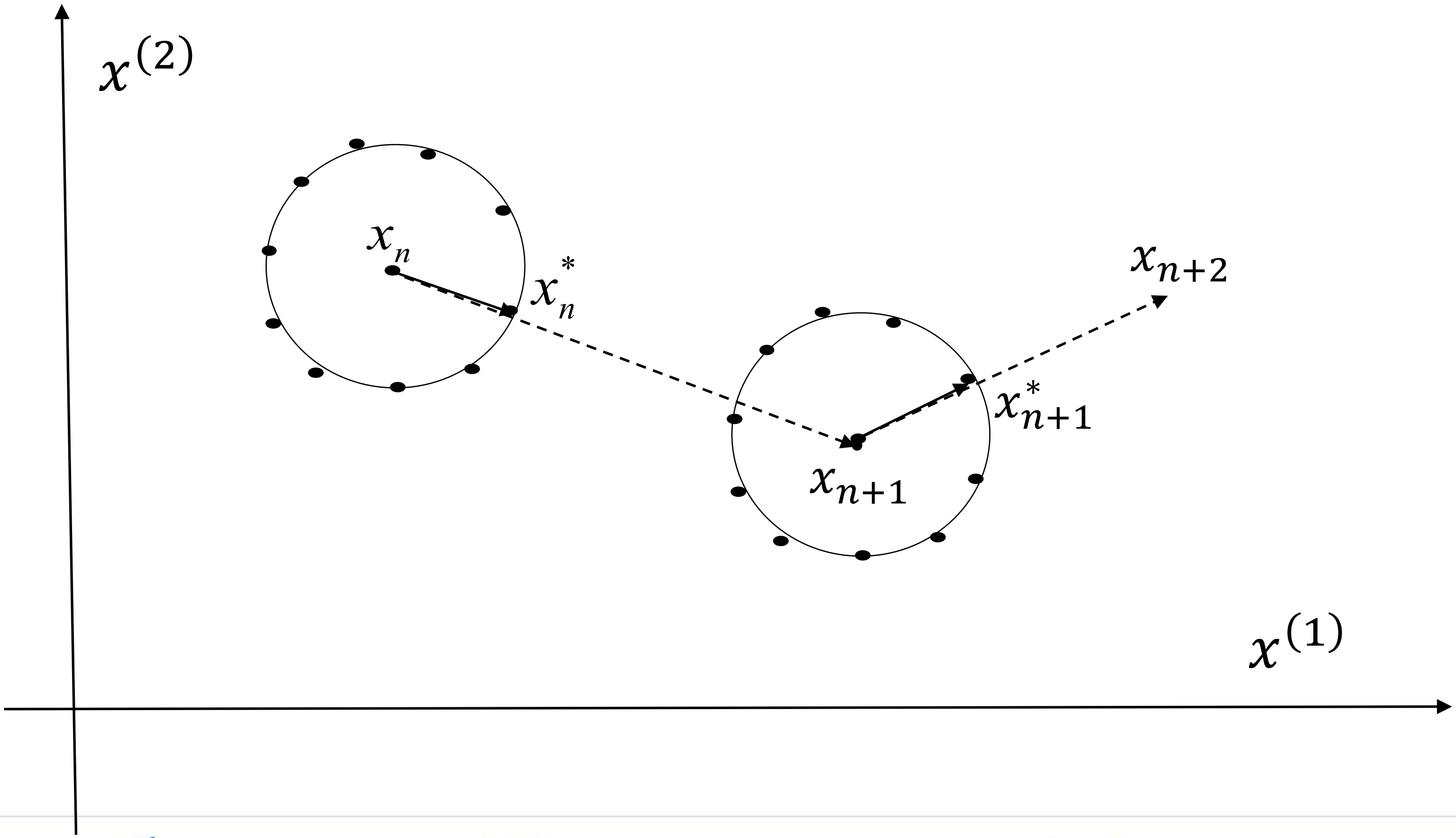
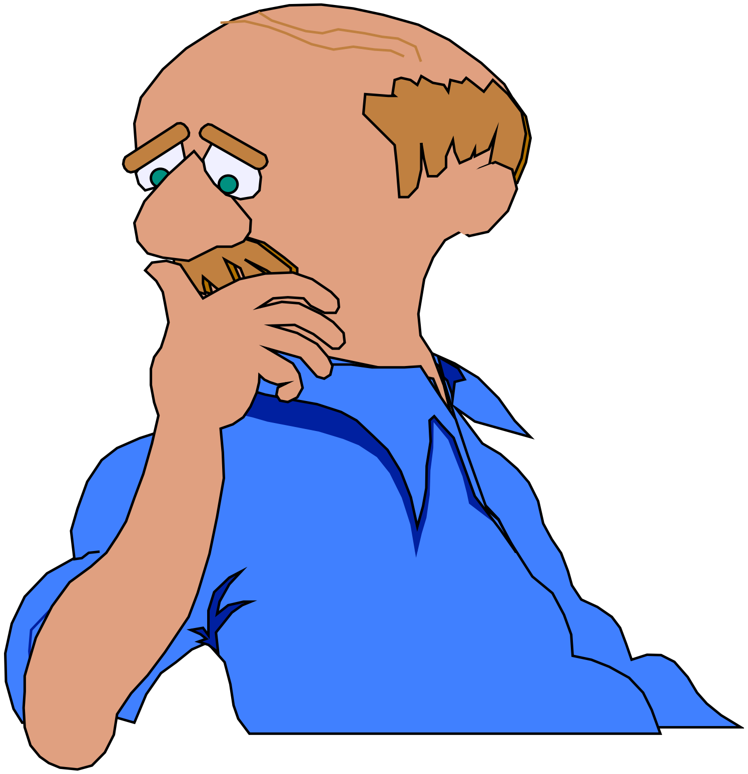

$x^{(2)}$
x_n
x_n^*
x_{n+2}
x_{n+1}^*
x_{n+1}
$x^{(1)}$

Thank you for attention



Particle Swarm Algorithm

- The goal of the algorithm is to gather all particles around the optimum of multidimensional hyperspace.
- Molecules are assigned a random position and a low initial velocity.
- The motion of molecules is simulated based on their velocity, position, best known solution, and global optimum.
- The molecules converge into the optimum/optima.

Algorytm Roju Cząstek

Celem algorytmu jest zgromadzenie wszystkich cząstek wokół optimum wielowymiarowej hiperprzestrzeni.

- Cząsteczkom przypisuje się losowe położenie i niską prędkość początkową.
- Symulowany jest ruch cząsteczek na podstawie jej prędkości, położenia, najlepszego znanego rozwiązania oraz optimum globalnego.
- Cząsteczki zbiegają do optimum/optimów.