Computer Science

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Systems Modelling and Analysis

Choose yourself and new technologies

L.16. Model based decision making

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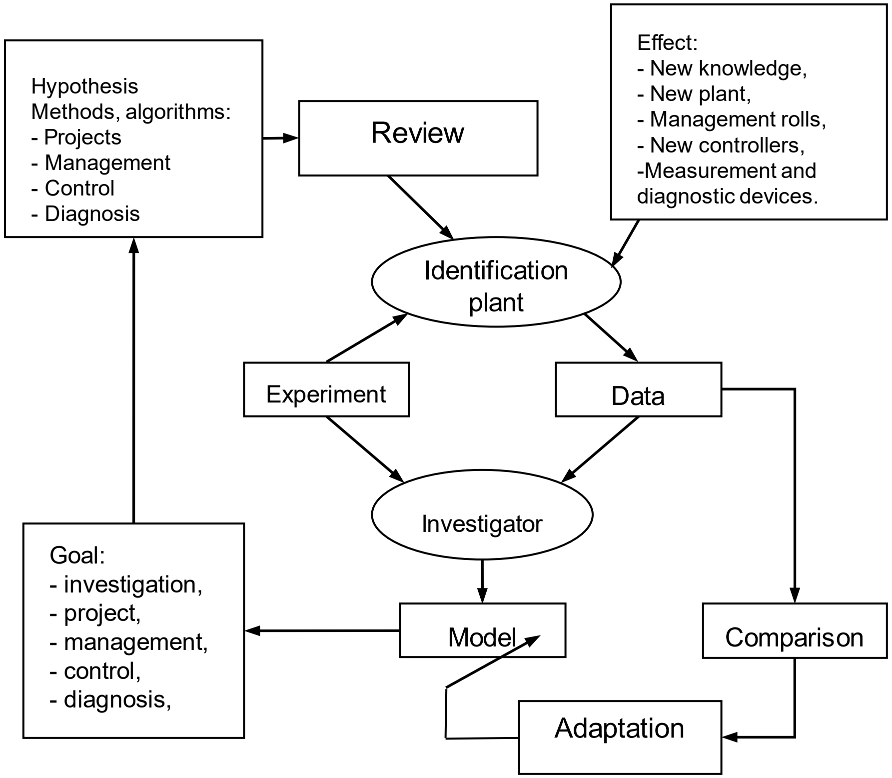
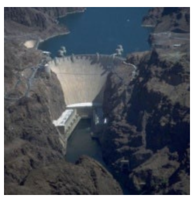
Model in the systems research



Example of decision making

Decision: workloads of power plants



hydroelectric plant

$x^{(1)}$

nuclear power plant

$x^{(2)}$

wind turbine

$x^{(3)}$

Images:
http://ziemianarozdrozu.pl/encyklopedia/67/hydroenergetyka
http://kresy24.pl/showNews/news_id/5871/
http://windy-future.info/2009/10/13/large-wind-turbine/

Given parameters:

c_1, c_2, c_3 – unit costs of workloads

Objective is to minimize overall costs: $F(x^{(1)}, x^{(2)}, x^{(3)}) = c_1 x^{(1)} + c_2 x^{(2)} + c_3 x^{(3)}$

Constraints: – demand must be met: $x^{(1)} + x^{(2)} + x^{(3)} \geq \beta$

– energy production capabilities are limited: $0 \leq x^{(n)} \leq \alpha_n, \quad n = 1,2,3$

Basic ingredients of optimization task formulation

Decision variables: $x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(S)} \end{bmatrix}$

Objective function: $y = F(x)$

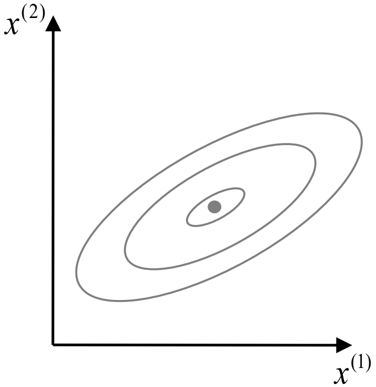
Set of feasible decisions (commonly defined by variables domain and constraints):

$$x \in \mathcal{D}_x$$

Optimization task: $x^* \rightarrow F(x^*) = \min_{x^* \in \mathcal{D}_x} F(x),$ x^* – optimal decision

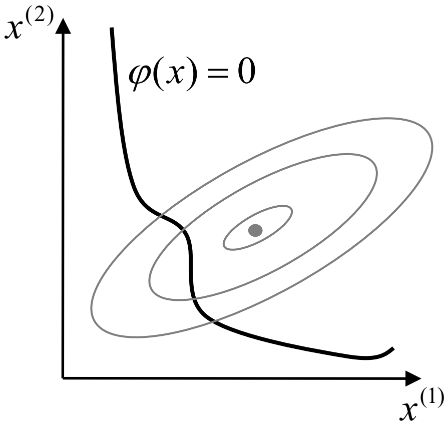
$$\min F(x) = -\max\big(-F(x)\big)$$

General classification of optimization tasks



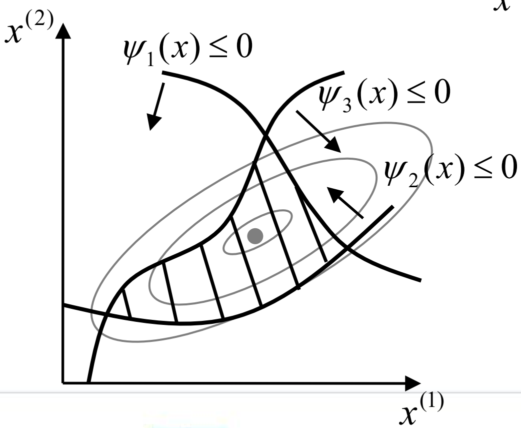
Unconstrained optimization:

$$\mathscr{D}_x = \mathscr{R}^S$$



Optimization under equality constraints:

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \varphi_1(x) = 0, \varphi_2(x) = 0, \ldots, \varphi_L(x) = 0, L \le S \right\}$$



Optimization under inequality constraints:

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \psi_1(x) \le 0, \psi_2(x) \le 0, \ldots, \psi_M(x) \le 0 \right\}$$

Analytical methods

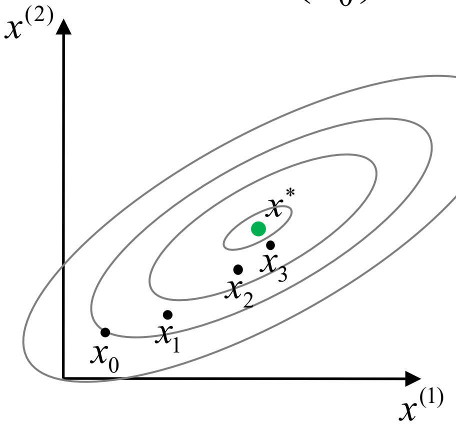
- Unconstrained optimization
- Lagrange multipliers method – equality constraints
- Kuhn-Tucker conditions – inequality constraints

Numerical methods

We only use information about values of objective function $F(x)$ for a given value of x.

$$x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_N \approx x^*$$

$$F(x_0) > F(x_1) > F(x_2) > \cdots > F(x_N) \approx F(x^*)$$



The general idea behind numerical methods.

Common types of optimization tasks

- Linear programming

Decision variables: $x \in \mathscr{D}_x \subseteq \mathscr{R}^S$

Objective function: $F(x) = c^T x = \sum_{s=1}^{S} c_s x^{(s)}$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_S \end{bmatrix}$$

Constraints:

$$\varphi_l(x) = a_l^T x - \alpha_l = 0,$$
$$l = 1,2,\ldots,L$$

$$a_l = \begin{bmatrix} a_l^{(1)} \\ a_l^{(2)} \\ \vdots \\ a_l^{(S)} \end{bmatrix}$$

$$\psi_m(x) = b_m^T x - \beta_m \le 0,$$
$$m = 1,2,\ldots,M$$

$$b_l = \begin{bmatrix} b_m^{(1)} \\ b_m^{(2)} \\ \vdots \\ b_m^{(S)} \end{bmatrix}$$

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Common types of optimization tasks

- Quadratic programming

Decision variables: $x \in \mathscr{D}_x \subseteq \mathscr{R}^S$

Objective function: $F(x) = x^T A x + b^T x + c$ $\qquad A \in \mathscr{R}^{S \times S},\ b \in \mathscr{R}^S,\ c \in \mathscr{R}$

Constraints:

$$\varphi_l(x) = d_l^T x - \alpha_l = 0, \quad l = 1,2,\ldots,L \qquad d_l = \begin{bmatrix} d_l^{(1)} \\ d_l^{(2)} \\ \vdots \\ d_l^{(S)} \end{bmatrix}$$

$$\psi_m(x) = e_m^T x - \beta_m \le 0, \quad m = 1,2,\ldots,M \qquad e_l = \begin{bmatrix} e_m^{(1)} \\ e_m^{(2)} \\ \vdots \\ e_m^{(S)} \end{bmatrix}$$

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Common types of optimization tasks

- Linear-fractional programming

Decision variables: $x \in \mathscr{D}_x \subseteq \mathscr{R}^S$

Objective function: $F(x) = \dfrac{a^T x + b}{c^T x + d}$ $\quad a \in \mathscr{R}^S,\ b \in \mathscr{R},\ c \in \mathscr{R}^S, d \in \mathscr{R}$

Constraints:

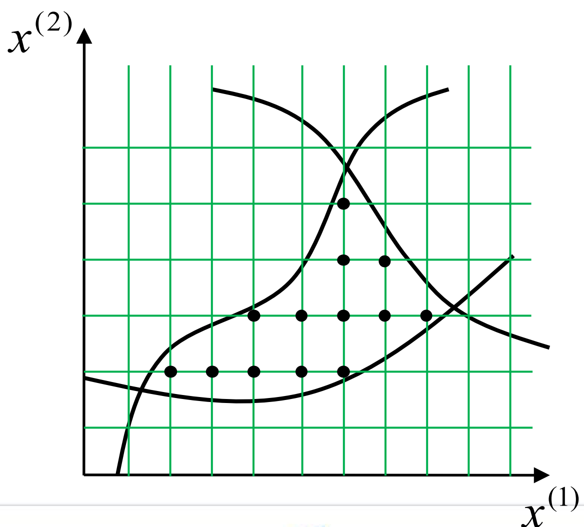
$$\varphi_l(x) = p_l^T x - \alpha_l = 0, \quad l = 1,2,\ldots,L \qquad p_l = \begin{bmatrix} p_l^{(1)} \\ p_l^{(2)} \\ \vdots \\ p_l^{(S)} \end{bmatrix}$$

$$\psi_m(x) = q_m^T x - \beta_m \le 0, \quad m = 1,2,\ldots,M \qquad q_l = \begin{bmatrix} q_m^{(1)} \\ q_m^{(2)} \\ \vdots \\ q_m^{(S)} \end{bmatrix}$$

Common types of optimization tasks

- Integer programming

Decision variables are discrete: $\overline{\mathcal{D}_x} = \mathcal{D}_x \cap \{x^{(s)} \in \mathscr{C}, s = 1,2,\ldots,S\}$

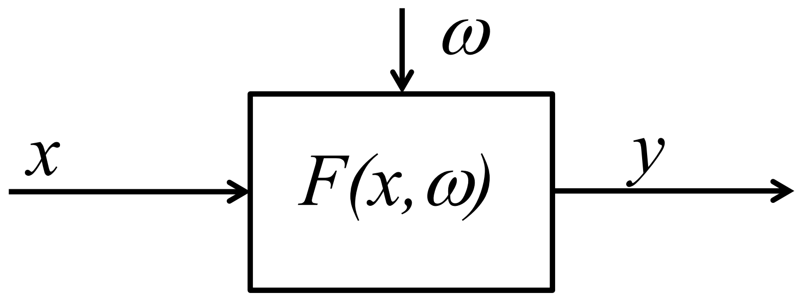


Special cases

$$x \in \overline{\mathcal{D}_x} = \{x_1, x_2, \ldots, x_M\}$$

$$x \in \overline{\mathcal{D}_x} = \{x^{(s)} \in \{0, 1\}, s = 1, 2, \ldots, S\}$$

Decision making under uncertainty



$$x^* \to F\left(x^*, \omega\right) = \min_{x \in D_x(\omega)} F\left(x, \omega\right) \quad ???$$

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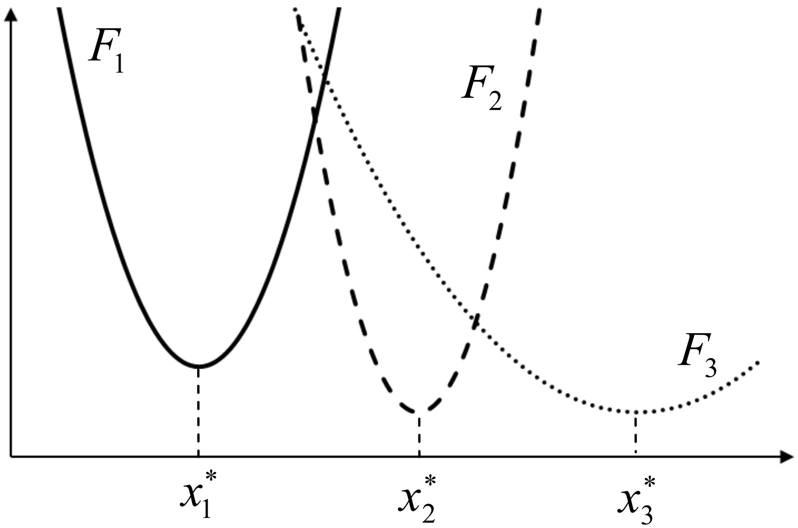
Multiobjective optimization

x – vector of decision variables

$F_1(x), F_2(x), \dots, F_M(x)$ – performance indices



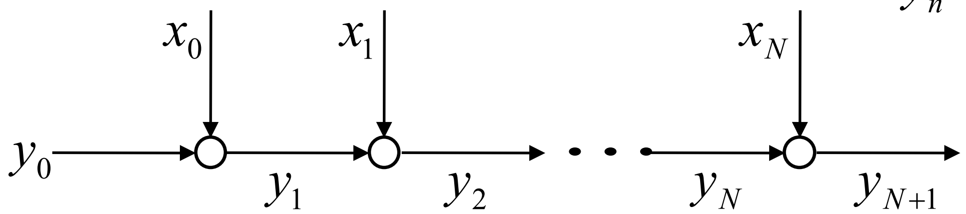
Dynamic optimization

Dynamic process: $y_{n+1} = P(y_n, x_n)$

n – time step

x_n – decision made at n-th time step

y_n – state of the process at n-th time step



http://www.all-freeware.com/

The proble is to find optimal sequence of decisions:

$$x_0^*, x_1^*, \ldots, x_N^*,$$

for which $Q(x_0, x_1, \ldots, x_N)$ is minimal.

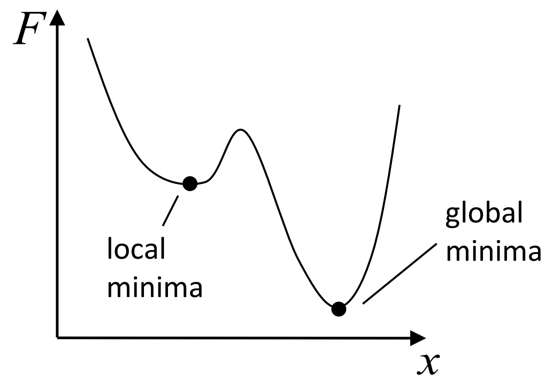
Mathematical preliminaries

Mathematical preliminaries

Optimization problem: $x^* \rightarrow F(x^*) = \min_{x \in \mathscr{D}_x} F(x)$

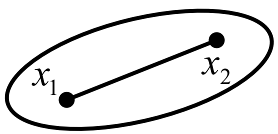
Local minima: $\forall_{\varepsilon > 0} \exists_{x \in O(x^*, \varepsilon)} F(x^*) < F(x)$

Global minima: $\forall_{x \in \mathscr{D}_x} F(x^*) < F(x)$

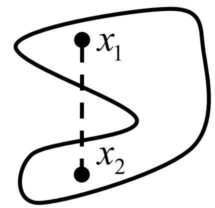


Mathematical preliminaries

Convex set: $\forall_{x_1, x_2 \in \mathscr{D}_x} \lambda x_1 + (1-\lambda) x_2 \in \mathscr{D}_x, \quad \lambda \in <0,1>$



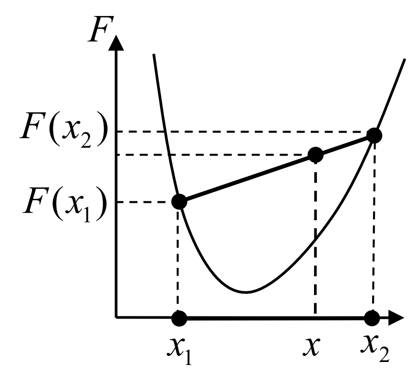
– convex set



– nonconvex set

Convex function:

$$F(\lambda x_1 + (1-\lambda) x_2) \le \lambda F(x_1) + (1-\lambda) F(x_2), \quad \lambda \in <0,1>$$



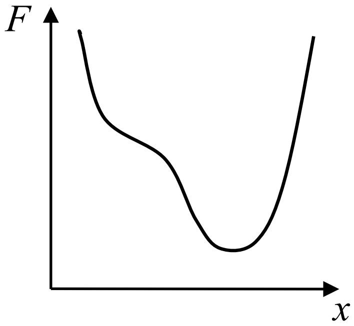
Mathematical preliminaries

Pseudo-convex function:

Following the Taylor's expansion of a function, we have:

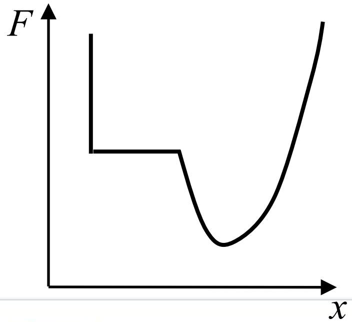
$$F(x) = F(x_0) + (x - x_0)^T \left[\nabla_x F(x_0)\right] + O_2\left(\|x - x_0\|\right)$$

$$(x - x_0)^T \left[\nabla_x F(x_0)\right] \geq 0 \quad \Rightarrow \quad F(x) > F(x_0)$$



Quasi-convex function:

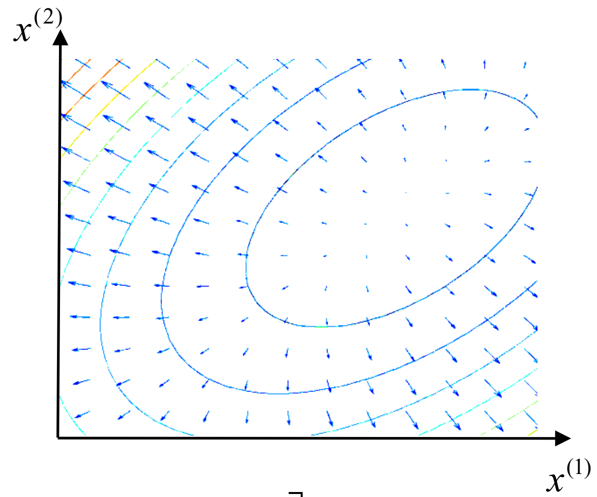
$$\mathcal{D}_\alpha = \left\{x \in \mathcal{D}_x : F(x) \leq \alpha\right\}$$ – convex sets



Mathematical preliminaries

Gradient:

$$\nabla_x F(x) = \begin{bmatrix} \dfrac{\partial F}{\partial x^{(1)}} \\ \dfrac{\partial F}{\partial x^{(2)}} \\ \vdots \\ \dfrac{\partial F}{\partial x^{(S)}} \end{bmatrix} = \underset{x}{grad}\, F(x)$$



Hessian:

$$H(x) = \nabla^2_{xx} F(x) = \begin{bmatrix} \dfrac{\partial^2 F}{\partial \left(x^{(1)}\right)^2} & \dfrac{\partial^2 F}{\partial x^{(1)} \partial x^{(2)}} & \cdots & \dfrac{\partial^2 F}{\partial x^{(1)} \partial x^{(S)}} \\ \dfrac{\partial^2 F}{\partial x^{(2)} \partial x^{(1)}} & \dfrac{\partial^2 F}{\partial \left(x^{(2)}\right)^2} & \cdots & \dfrac{\partial^2 F}{\partial x^{(2)} \partial x^{(S)}} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 F}{\partial x^{(S)} \partial x^{(1)}} & \dfrac{\partial^2 F}{\partial x^{(S)} \partial x^{(2)}} & \cdots & \dfrac{\partial^2 F}{\partial \left(x^{(S)}\right)^2} \end{bmatrix}$$

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Mathematical preliminaries

Hessian properties:

$$\frac{\partial^2 F}{\partial x^{(i)} \partial x^{(j)}} = \frac{\partial^2 F}{\partial x^{(j)} \partial x^{(i)}} \Rightarrow H$$ is symmetric matrix

If $\forall_{x \neq 0_S} x^T H x > 0$ then H is positive definite

If $\forall_{x \neq 0_S} x^T H x < 0$ then H is negative definite

If $\forall_{x \neq 0_S} x^T H x \geq 0$ then H is positive semidefinite

If $\forall_{x \neq 0_S} x^T H x \leq 0$ then H is negative semidefinite

Sylwester criteria:

$$H = \left[h_{ij}\right]_{\substack{i=1,2,\dots,S \\ j=1,2,\dots,S}}$$ - Hess matrix

If $\forall\, s = 1, 2, \dots, S \quad \det\left(H_{ss}\right) = \det\left(\left[h_{ij}\right]_{\substack{i=1,2,\dots,s \\ j=1,2,\dots,s}}\right) > 0$ then matrix *H* is positive definite

if $\forall\, \{i_1, i_2, \cdots, i_s\} \in \{1, 2, \cdots, S\} \quad \det\left(\left[h_{ij}\right]_{\substack{i\in\{i_1,i_2,\cdots,i_s\} \\ j\in\{i_1,i_2,\cdots,i_s\}}}\right) \geq 0$ then matrix *H* is semipositive definite

Eigen values of matrix *H*

$\det\left(H - hI\right) = 0 \quad h_1, h_2, \dots, h_S$ - Eigen values of matrix *H*

If $\forall\, s = 1, 2, \dots, S \quad h_s > 0$ then matrix *H* is positive definite

If $\forall\, s = 1, 2, \dots, S \quad h_s \geq 0$ then matrix *H* is semipositive definite

$$\forall\, s = 1, 2, \ldots, S \quad \det(H_{ss}) = \det\left([h_{ij}]_{\substack{i=1,2,\ldots,s \\ j=1,2,\ldots,s}} \right) > 0$$

$$H = [h_{ij}]_{\substack{i=1,2,\cdots,S \\ i=1,2,\cdots,S}} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1S} \\ h_{21} & h_{22} & h_{23} & \cdots & h_{2S} \\ h_{31} & h_{32} & h_{33} & \cdots & h_{3S} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{S1} & h_{S2} & h_{S3} & \cdots & h_{SS} \end{bmatrix}$$

$$\forall \{i_1, i_2, \cdots, i_s\} \in \{1, 2, \cdots, S\} \quad \det\left(\left[h_{ij}\right]_{\substack{i\in\{i_1,i_2,\cdots,i_s\} \\ j\in\{i_1,i_2,\cdots,i_s\}}} \right) \geq 0$$

For example $\{i_1, i_2, i_2\} = \{1,3,7\}$

$$det\left(\left[h_{ij}\right]_{\substack{i=1,3,S \\ i=1,3,S}}\right) = det\begin{bmatrix} h_{11} & h_{13} & h_{1S} \\ h_{31} & h_{33} & h_{3S} \\ h_{S1} & h_{S3} & h_{SS} \end{bmatrix} \geq 0$$

$$H = \left[h_{ij}\right]_{\substack{i=1,2,\cdots,S \\ i=1,2,\cdots,S}} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1S} \\ h_{21} & h_{22} & h_{23} & \cdots & h_{2S} \\ h_{31} & h_{32} & h_{33} & \cdots & h_{3S} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{S1} & h_{S2} & h_{S3} & \cdots & h_{SS} \end{bmatrix}$$

$$\begin{bmatrix} \boldsymbol{h_{11}} & h_{12} & \boldsymbol{h_{13}} & \cdots & \boldsymbol{h_{1S}} \\ h_{21} & h_{22} & h_{23} & \cdots & h_{2S} \\ \boldsymbol{h_{31}} & h_{32} & \boldsymbol{h_{33}} & \cdots & \boldsymbol{h_{3S}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{h_{S1}} & h_{S2} & \boldsymbol{h_{S3}} & \cdots & \boldsymbol{h_{SS}} \end{bmatrix}$$

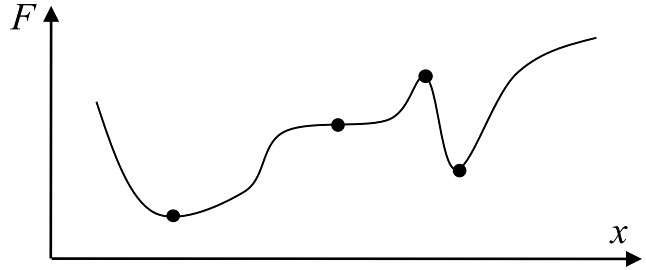
Unconstrained optimization

Optimization task: $x^* \rightarrow F(x^*) = \min_{x^* \in \mathcal{D}_x} F(x)$

Assumption: $F(x)$ is continuous and differentiable.

Necessary condition for x^* to be local minima: $\nabla_x F(x^*) = 0_S$

If $F(x)$ is convex function, then above equation is sufficient condition for x^* to be global minima.



Unconstrained optimization

Second order conditions of optimality:

If $H(x^*)$ is positive definite at x^* then x^* is local minimum.

If $H(x^*)$ is negative definite at x^* then x^* is local maximum.

If $H(x^*)$ is neither negative semidefinite nor positive semidefinite at x^* then x^* is not optimum.

If $H(x^*)$ is positive (negative) semidefinite and not positive (negative) definite, optimality of x^* cannot be determined.

Example 2.1.1

- $F(x^{(1)}, x^{(2)}) = 5(x^{(1)})^2 + (x^{(2)})^2 - 4x^{(1)}x^{(2)} - 2x^{(1)} + 3$
- $\nabla_x F(x^{(1)}, x^{(2)})\big|_{x=x^*} = \left.\begin{bmatrix} \frac{\partial F(x^{(1)},x^{(2)})}{\partial x^{(1)}} \\ \frac{\partial F(x^{(1)},x^{(2)})}{\partial x^{(2)}} \end{bmatrix}\right|_{x=x^*} = \begin{bmatrix} 10x^{(1)*} - 4x^{(2)*} - 2 \\ 2x^{(2)*} - 4x^{(1)*} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix}(1)\\(2)\end{matrix}$
- z $(2) \rightarrow x^{(2)*} = 2x^{(1)*}$
- z $(1) \rightarrow 10x^{(1)*} - 8x^{(1)*} = 2 \rightarrow x^{(1)*} = 1, x^{(2)*} = 2$ $\quad x^* = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
- $H(x) = \nabla_{xx} F(x^{(1)}, x^{(2)}) = \begin{bmatrix} 10 & -4 \\ -4 & 2 \end{bmatrix}$
- $detH_{11} = det[10] = 10 > 0$
- $detH_{11} = det \begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix} = 20 - 16 = 4 > 0$

Matrix H(x) is positively defined then point X* is minimum

- Macierz $H(x)$ jest dodatnio określona zatem punkt $x^* = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ - minimum

Example 2.1.2

- $F(x^{(1)},x^{(2)}) = \alpha(x^{(1)})^2 + (x^{(2)})^2 - 4x^{(1)}x^{(2)} - 2x^{(1)} + 3$
- $\nabla_x F(x^{(1)},x^{(2)})\big|_{x=x^*} = \left[\begin{matrix} \frac{\partial F(x^{(1)},x^{(2)})}{\partial x^{(1)}} \\ \frac{\partial F(x^{(1)},x^{(2)})}{\partial x^{(2)}} \end{matrix}\right]\Bigg|_{x=x^*} = \begin{bmatrix} 2\alpha x^{(1)*} - 4x^{(2)*} - 2 \\ 2x^{(2)*} - 4x^{(1)*} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix}(1)\\(2)\end{matrix}$
- z $(2) \rightarrow x^{(2)*} = 2x^{(1)*}$
- z $(1) \rightarrow 2\alpha x^{(1)*} - 8x^{(1)*} = 2 \rightarrow x^{(1)*} = \frac{1}{\alpha-4}, x^{(2)*} = \frac{2}{\alpha-4}, \alpha \neq 4$ $\quad x^* = \begin{bmatrix} \frac{1}{\alpha-4} \\ \frac{2}{\alpha-4} \end{bmatrix}$
- $H(x) = \nabla_{xx} F(x^{(1)},x^{(2)}) = \begin{bmatrix} 2\alpha & -4 \\ -4 & 2 \end{bmatrix}$
- $detH_{11} = det[2\alpha] = 2\alpha > 0 \rightarrow \alpha > 0$
- $detH_{11} = det\begin{bmatrix} 2\alpha & 4 \\ 4 & 2 \end{bmatrix} = 4\alpha - 16 > 0 \rightarrow \alpha > 4$

For $\alpha > 4$ matrix H(x) is positively defined then point $\begin{bmatrix} \frac{1}{\alpha-4} \\ \frac{2}{\alpha-4} \end{bmatrix}$ - minimum

- Dla $\alpha > 4$ macierz $H(x)$ jest dodatnio określona a punkt $x^* = \begin{bmatrix} \frac{1}{\alpha-4} \\ \frac{2}{\alpha-4} \end{bmatrix}$ - minimum

Example 2.1.2 c.d.

$For\ \alpha > 4\ matrix\ H(x)\ is\ positively\ defined$

$$Point\ x^* = \begin{bmatrix} \frac{1}{\alpha - 4} \\ \frac{2}{\alpha - 4} \end{bmatrix}$$ is minimum $(\alpha \neq 4)$

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Example 2.1.3

- $F(x) = x^T Ax + b^T x + c$
- A – macierz symetryczna, dodatnio określona
  A – symmetric matrix, positive define
- $A = [a_{ij}]_{\substack{i=1,2,\cdots,S\\ i=1,2,\cdots,S}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1S} \\ a_{21} & a_{22} & \cdots & a_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ a_{S1} & a_{S2} & \cdots & a_{SS} \end{bmatrix}$
- $x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(S)} \end{bmatrix}$, b= $\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_S \end{bmatrix}$, - S –wymiarowe wektory
  S – dimensional vectors
- $\nabla_x F(x)|_{x=x^*} = \nabla_x(x^T Ax + b^T x + c\,)|_{x=x^*} = 0_S \qquad |_{x=x^*}$

Example 2.1.3 c.d.

- $x^T A x = \begin{bmatrix} x^{(1)} & x^{(2)} & \cdots & x^{(S)} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1S} \\ a_{21} & a_{22} & \cdots & a_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ a_{S1} & a_{S2} & \cdots & a_{SS} \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(S)} \end{bmatrix}$

- $x^T A x = \sum_{i=1}^{S} \sum_{j=1}^{S} a_{ij}\, x^{(i)} x^{(j)}$

- $\nabla_x (x^T A x) = \begin{bmatrix} \frac{\partial (x^T A x)}{\partial x^{(1)}} \\ \frac{\partial (x^T A x)}{\partial x^{(2)}} \\ \vdots \\ \frac{\partial (x^T A x)}{\partial x^{(S)}} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{x^{(1)}} \left(\sum_{i=1}^{S} \sum_{j=1}^{S} a_{ij}\, x^{(i)} x^{(j)} \right) \\ \frac{\partial}{x^{(2)}} \left(\sum_{i=1}^{S} \sum_{j=1}^{S} a_{ij}\, x^{(i)} x^{(j)} \right) \\ \vdots \\ \frac{\partial}{x^{(S)}} \left(\sum_{i=1}^{S} \sum_{j=1}^{S} a_{ij}\, x^{(i)} x^{(j)} \right) \end{bmatrix}$

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Example 2.1.3 c.d.

- $$\nabla_x (x^T A x) = \begin{bmatrix} \frac{\partial}{\partial x^{(1)}} \left(\sum_{i=1}^{S} \sum_{j=1}^{S} a_{ij} \, x^{(i)} x^{(j)} \right) \\ \frac{\partial}{\partial x^{(2)}} \left(\sum_{i=1}^{S} \sum_{j=1}^{S} a_{ij} \, x^{(i)} x^{(j)} \right) \\ \vdots \\ \frac{\partial}{\partial x^{(S)}} \left(\sum_{i=1}^{S} \sum_{j=1}^{S} a_{ij} \, x^{(i)} x^{(j)} \right) \end{bmatrix}$$

- $$\nabla_x (x^T A x) = \begin{bmatrix} \sum_{j=1}^{S} a_{1j} x^{(j)} + \sum_{i=1}^{S} a_{i1} x^{(i)} \\ \sum_{j=1}^{S} a_{2j} x^{(j)} + \sum_{i=1}^{S} a_{i2} x^{(i)} \\ \vdots \end{bmatrix} =$$

Example 2.1.3 c.d.

- $$\nabla_x(x^T A x) = \begin{bmatrix} \sum_{j=1}^{S} a_{1j} x^{(j)} \\ \sum_{j=1}^{S} a_{2j} x^{(j)} \\ \vdots \\ \sum_{j=1}^{S} a_{Sj} x^{(j)} \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{S} a_{i1} x^{(i)} \\ \sum_{i=1}^{S} a_{i2} x^{(i)} \\ \vdots \\ \sum_{i=1}^{S} a_{S1} x^{(i)} \end{bmatrix}$$

$$\nabla_x(x^T A x) =$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1S} \\ a_{21} & a_{22} & \cdots & a_{2S} \\ \vdots & \vdots & \ddots & \vdots \\ a_{S1} & a_{S2} & \cdots & a_{SS} \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(S)} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{S1} \\ a_{12} & a_{22} & \cdots & a_{S2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{S1} & a_{S2} & \cdots & a_{SS} \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(S)} \end{bmatrix}$$

$$\nabla_x(x^T A x) = Ax + A^T x$$

$$dla\ A = A^T \qquad \nabla_x(x^T A x) = 2Ax$$

Example 2.1.3 c.d.

- $$b^T x = [b_1 \quad b_2 \quad \cdots \quad b_S] \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(S)} \end{bmatrix} = \sum_{i=1}^{S} b_i x^{(j)}$$

- $$\nabla_x (b^T x) = \begin{bmatrix} \frac{\partial}{\partial x^{(1)}} \left(\sum_{i=1}^{S} b_i x^{(j)} \right) \\ \frac{\partial}{\partial x^{(2)}} \left(\sum_{i=1}^{S} b_i x^{(j)} \right) \\ \vdots \\ \frac{\partial}{\partial x^{(S)}} \left(\sum_{i=1}^{S} b_i x^{(j)} \right) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_S \end{bmatrix} = \mathrm{b}$$

Example 2.1.3 c.d.

- $\nabla_x (x^T A x + b^T x + c)|_{x=x^*} = 2\mathrm{A}x^* + \mathrm{b} = 0_S$
- $x^* = -\frac{1}{2} A^{-1} b$
- $H(x) = \nabla_{xx} (x^T A x + b^T x + c) = \nabla_x (2\mathrm{Ax} + \mathrm{b}) = 2A$
- Macierz Hessa dodatnio określona bo A jest dodatnio określona

Hess matrix is positively defined because
matrix A is assumed to be positively defined

Basic formulation of the optimization problem

Decision variables: $x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(S)} \end{bmatrix}$

Objective function: $y = F(x)$

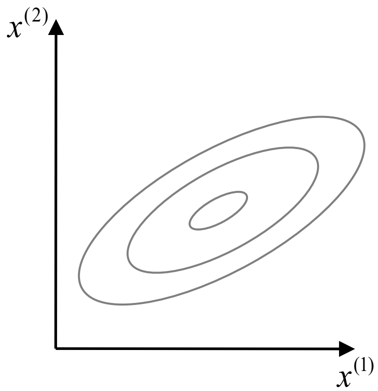
Set of feasible decisions (commonly defined by variables domain and constraints):

$$x \in \mathcal{D}_x$$

Optimization task: $x^* \to F(x^*) = \min_{x^* \in \mathcal{D}_x} F(x),$ x^* – optimal decision

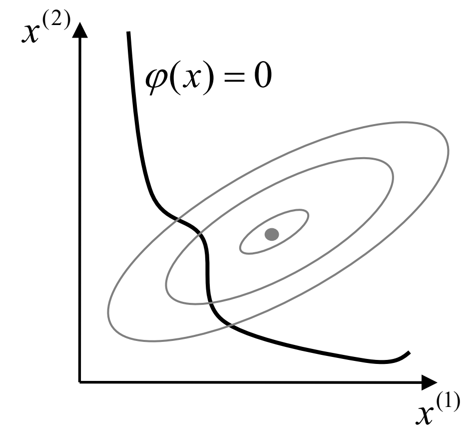
$$\min F(x) = -\max\big(-F(x)\big)$$

General classification of optimization tasks



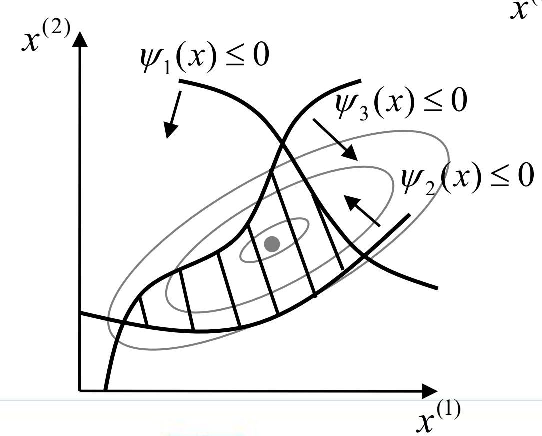
Unconstrained optimization:

$$\mathscr{D}_x = \mathscr{R}^S$$



Optimization under equality constraints:

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \varphi_1(x) = 0, \varphi_2(x) = 0, \ldots, \varphi_L(x) = 0, L \leq S \right\}$$



Optimization under inequality constraints:

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \psi_1(x) \leq 0, \psi_2(x) \leq 0, \ldots, \psi_M(x) \leq 0 \right\}$$

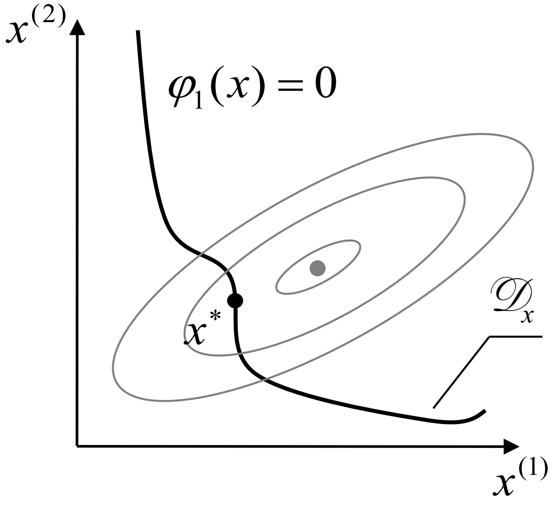
Analytical methods

- Unconstrained optimization
- Lagrange multipliers method – equality constraints
- Kuhn-Tucker conditions – inequality constraints

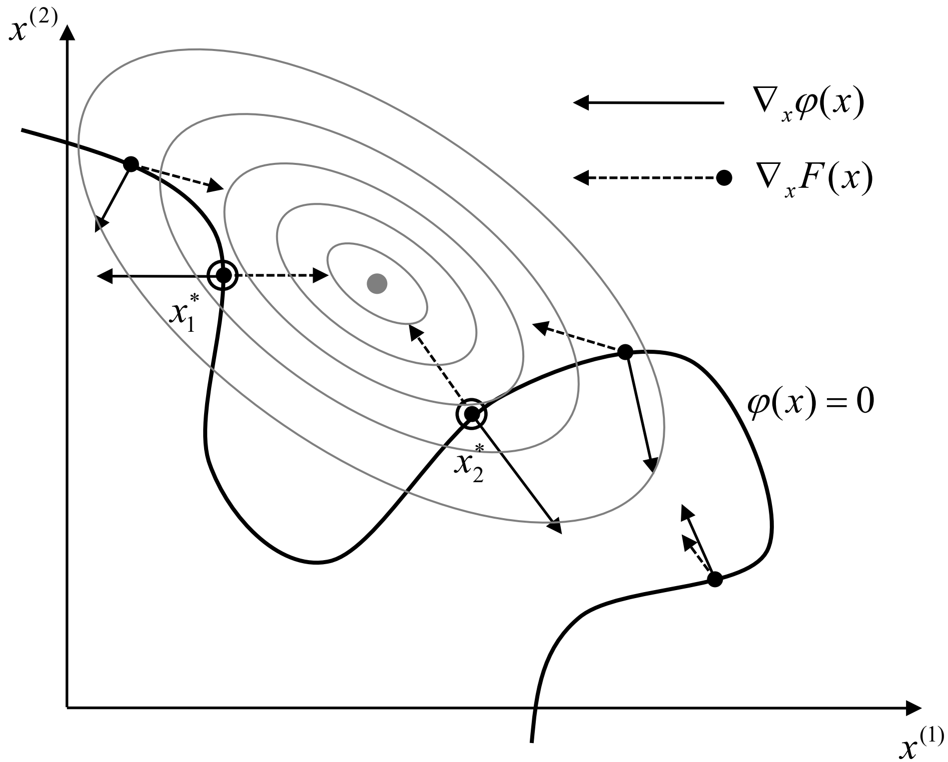
Optimization under equality constraints

Optimization task: $x^* \to F(x^*) = \min_{x^* \in \mathscr{D}_x} F(x)$

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \ \varphi_1(x) = 0, \varphi_2(x) = 0, \ldots, \varphi_L(x) = 0, \ L \le S \right\}$$



Optimization under equality constraints



Locally optimal solution satisfies condition:

$$\nabla_x F(x) + \lambda \nabla_x \varphi(x) = 0_S$$

where

$\lambda \in \mathscr{R}$ – Lagrange multiplier

For multiple constraints:

$$\nabla_x F(x) + \sum_{l=1}^{L} \lambda_l \nabla_x \varphi_l(x) = 0_S$$

Optimization under equality constraints

- The method of Lagrange multipliers

Lagrange function:

$$L(x,\lambda) = F(x) + \sum_{l=1}^{L} \lambda_l \varphi_l(x) = F(x) + \lambda^T \varphi(x)$$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_L \end{bmatrix}, \quad \varphi(x) = \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \\ \varphi_L(x) \end{bmatrix}$$

Necessary conditions of optimality:

$$\nabla_x L(x,\lambda)\big|_{x^*,\lambda^*} = 0_S$$

$$\nabla_\lambda L(x,\lambda)\big|_{x^*,\lambda^*} = 0_L$$

If and only if $\operatorname{rank} G(x) = \operatorname{rank}\left[G(x) \;\vdots\; -\nabla_x F(x)\right]$,

Where: $G(x) = \left[\nabla_x \varphi_1(x) \;\vdots\; \nabla_x \varphi_2(x) \;\vdots\; \cdots \;\vdots\; \nabla_x \varphi_L(x)\right]$

Optimization problem under equality constraints
Lagrange' a multiplayers metod

The above system of equation may have several solutions

Second order nesesery conditions:

let: $$H_L(x) = \nabla_{xx} L(x, \lambda)$$

If $H_L(x^*)$ Is positively defined in the point x^*
then x^* is local minimum

If $H_L(x^*)$ Is negatively defined in the point x^*
then x^* is local minimum

If $F(x)$ is convex function, and constrains are linear one i.e. have the form
$\varphi_l(x) = p_l^T x - \alpha_l = 0, \quad l = 1,2,\ldots,L$ then the above system of equation
have one solution and it is optimsl point

Explanation of necessary conditions

- $x_1^*, x_2^* \rightarrow F(x_1^*, x_2^*) = \min_{x_1, x_2} F(x_1, x_2)$

With constans

- Przy ograniczeniu $\varphi(x_1, x_2) = 0$
- $\varphi(x_1, x_2) = 0 \rightarrow x_2 = \psi(x_1)$
- $x_1^* \rightarrow F\big(x_1^*, \psi(x_1^*)\big) = \min_{x_1} F\big(x_1 \psi(x_1)\big)$
- $\frac{dF\big(x_1 \psi(x_1)\big)}{\partial x_1} = \frac{\partial F(x_1, x_2)}{\partial x_1} + \frac{\partial F(x_1, x_2)}{\partial x_2} \frac{d\psi(x_1)}{dx_1} = 0$
- Pochodna funkcji rozwikłanej
- $\frac{d\psi(x_1)}{\partial x_1} = -\frac{\frac{\partial \varphi(x_1, x_2)}{\partial x_1}}{\frac{\partial \varphi(x_1, x_2)}{\partial x_2}}$

Derived from the unraveling function

Explanation of necessary conditions

- $$\frac{\partial F(x_1,x_2)}{\partial x_1}+\frac{\partial F(x_1,x_2)}{\partial x_2}\left(-\frac{\frac{\partial \varphi(x_1,x_2)}{\partial x_1}}{\frac{\partial \varphi(x_1,x_2)}{\partial x_2}}\right)=0$$

let

- oznaczmy $\lambda = -\frac{\frac{\partial F(x_1,x_2)}{\partial x_2}}{\frac{\partial \varphi(x_1,x_2)}{\partial x_2}}$
- $$\frac{\partial F(x_1,x_2)}{\partial x_1}+\lambda\frac{\partial \varphi(x_1,x_2)}{\partial x_1}=0$$
- $$\frac{\partial F(x_1,x_2)}{\partial x_2}+\lambda\frac{\partial \varphi(x_1,x_2)}{\partial x_2}=0$$
- $x_2=\psi(x_1)\rightarrow\varphi(x_1,x_2)=0$

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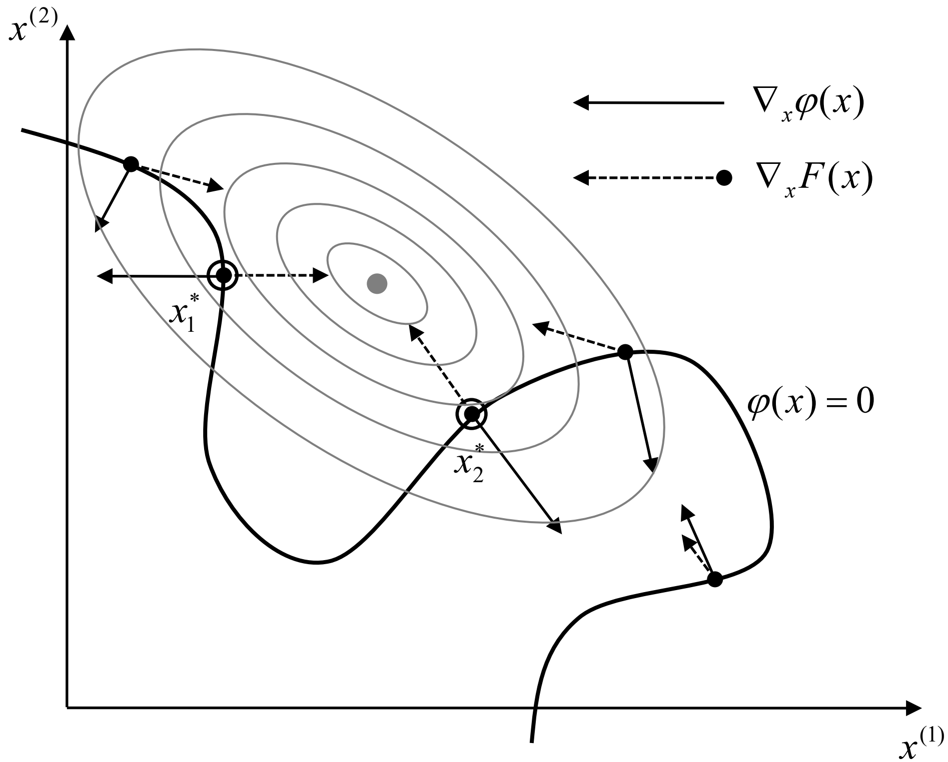
Explanation of necessary conditions

- $L(x_1, x_2, \lambda) = F(x_1, x_2) + \lambda\varphi(x_1, x_2)$
- $\frac{\partial L(x_1,x_2,\lambda)}{\partial x_1} = 0 \to \frac{\partial F(x_1,x_2)}{\partial x_1} + \lambda \frac{\partial \varphi(x_1,x_2)}{\partial x_1} = 0$
- $\frac{\partial L(x_1,x_2,\lambda)}{\partial x_2} = 0 \to \frac{\partial F(x_1,x_2)}{\partial x_2} + \lambda \frac{\partial \varphi(x_1,x_2)}{\partial x_2} = 0$
- $\frac{\partial L(x_1,x_2,\lambda)}{\partial \lambda} = 0 \to \varphi(x_1, x_2) = 0$
- Ogólniej More general

$$\nabla_x L(x, \lambda)\big|_{x^*, \lambda^*} = 0_S$$

$$\nabla_\lambda L(x, \lambda)\big|_{x^*, \lambda^*} = 0_L$$

Optimization under equality constraints



Locally optimal solution satisfies condition:

$$\nabla_x F(x) + \lambda \nabla_x \varphi(x) = 0_S$$

where

$\lambda \in \mathscr{R}$ – Lagrange multiplier

For multiple constraints:

$$\nabla_x F(x) + \sum_{l=1}^{L} \lambda_l \nabla_x \varphi_l(x) = 0_S$$

$$\nabla_x L(x, \lambda)\Big|_{x^*, \lambda^*} = 0_S$$

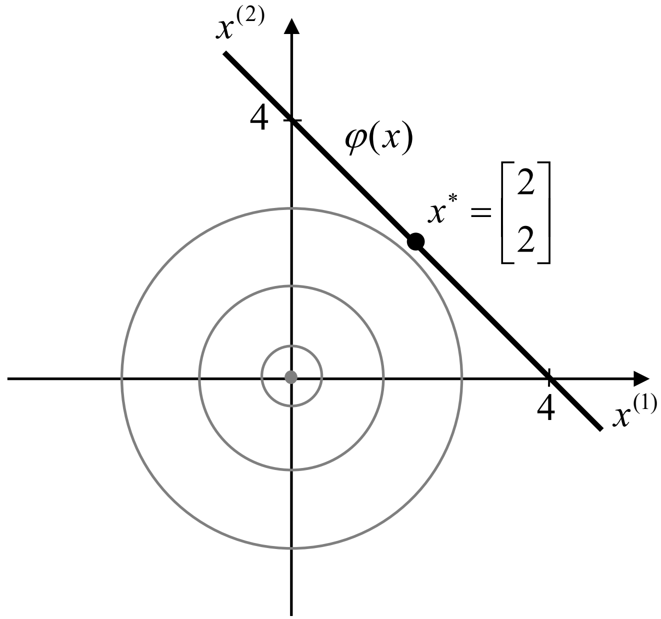
Optimization under equality constraints

- The method of Lagrange multipliers – example 1

$$F(x) = \left(x^{(1)}\right)^2 + \left(x^{(2)}\right)^2$$

$$\varphi(x) = x^{(1)} + x^{(2)} - 4 = 0$$

$$L(x,\lambda) = \left(x^{(1)}\right)^2 + \left(x^{(2)}\right)^2 + \lambda\left(x^{(1)} + x^{(1)} - 4\right)$$



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Example 2.2.1

- $L(x,\lambda) = \left(x^{(1)}\right)^2 + \left(x^{(2)}\right)^2 + \lambda\left(x^{(1)} + x^{(2)} - 4\right)$
- $\nabla_x L(x,\lambda) = \begin{bmatrix} \frac{\partial L}{\partial x^{(1)}} \\ \frac{\partial L}{\partial x^{(2)}} \end{bmatrix} = \begin{bmatrix} 2x^{(1)} + \lambda \\ 2x^{(2)} + \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} (1) \\ (2) \end{matrix}$
- $\nabla_\lambda L(x,\lambda) = \frac{\partial L}{\partial \lambda} = x^{(1)} + x^{(2)} - 4 = 0 \qquad (3)$
- z $(1) \rightarrow x^{(1)} = -\frac{\lambda}{2}$, z $(2) \rightarrow x^{(2)} = -\frac{\lambda}{2}$
- z $(3) \rightarrow \left(-\frac{\lambda}{2}\right) + \left(-\frac{\lambda}{2}\right) - 4 = 0$ czyli $\lambda = -4$
- $x^{(1)} = -\frac{\lambda}{2} = -\frac{-4}{2} = 2$, $x^{(2)} = -\frac{\lambda}{2} = -\frac{-4}{2} = 2$

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Example 2.2.1 c.d.

- $\nabla_x L(x,\lambda) = \begin{bmatrix} \frac{\partial L}{\partial x^{(1)}} \\ \frac{\partial L}{\partial x^{(2)}} \end{bmatrix} = \begin{bmatrix} 2x^{(1)} + \lambda \\ 2x^{(2)} + \lambda \end{bmatrix}$
- $H_L = \nabla_{xx} L(x,\lambda) = \begin{bmatrix} \frac{\partial^2 L}{\partial^2 x^{(1)}} & \frac{\partial^2 L}{\partial x^{(1)} \partial x^{(2)}} \\ \frac{\partial^2 L}{\partial x^{(2)} \partial x^{(1)}} & \frac{\partial^2 L}{\partial^2 x^{(2)}} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$
- $det\, H_{L11} = det[2] = 2 > 0, \quad det\, H_{L22} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2 \times 2 = 4 > 0$

Matrix — Is positively defined
- Macierz $H_L = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ jest dodatnio określna

Point — is minimum
- Punkt $x = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ - minimum

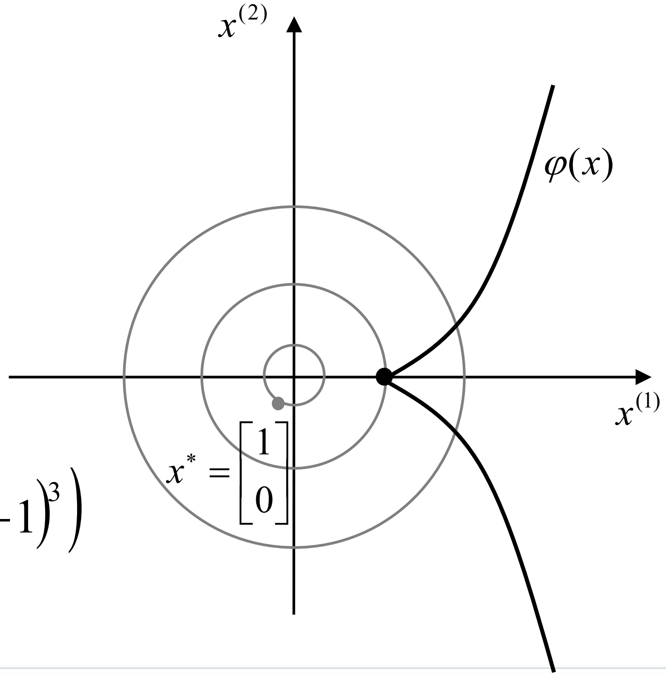
Optimization under equality constraints

- The method of Lagrange multipliers – example 2 (irregular)

$$F(x) = \left(x^{(1)}\right)^2 + \left(x^{(2)}\right)^2$$

$$\varphi(x) = \left(x^{(2)}\right)^2 - \left(x^{(1)} - 1\right)^3 = 0$$

$$L(x,\lambda) = \left(x^{(1)}\right)^2 + \left(x^{(2)}\right)^2 + \lambda\left(\left(x^{(2)}\right)^2 - \left(x^{(1)} - 1\right)^3\right)$$



Example 2.2.2

- $L(x,\lambda) = \left(x^{(1)}\right)^2 + \left(x^{(2)}\right)^2 + \lambda\left(\left(x^{(2)}\right)^2 - \left(x^{(1)} - 1\right)^3\right)$
- $\nabla_x L(x,\lambda) = \begin{bmatrix} \frac{\partial L}{\partial x^{(1)}} \\ \frac{\partial L}{\partial x^{(2)}} \end{bmatrix} = \begin{bmatrix} 2x^{(1)} - 3\lambda\left(x^{(1)} - 1\right)^2 \\ 2x^{(2)} + 2\lambda x^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} (1) \\ (2) \end{matrix}$
- $\nabla_\lambda L(x,\lambda) = \frac{\partial L}{\partial \lambda} = \left(x^{(2)}\right)^2 - \left(x^{(1)} - 1\right)^3 = 0 \qquad (3)$
- z $(2) \rightarrow 2(1+\lambda)x^{(2)} = 0$ czyli $x^{(2)} = 0,$
- z $(3) \rightarrow (0)^2 - \left(x^{(1)} - 1\right)^3 = 0,$ czyli $x^{(1)} = 1,$
- z $(1) \rightarrow 2x^{(1)} - 3\lambda\left(x^{(1)} - 1\right)^2 = 2\text{x}1 - 3\lambda(1-1)^2 = 2 \neq 0$
- Sprzeczność ?? Contradiction

Optimization under equality constraints

- The method of Lagrange multipliers – example 2 explanation

$$\nabla_x L(x,\lambda) = \nabla_x F(x) + \sum_{l=1}^{L} \lambda_l \nabla_x \varphi_l(x) = 0_S$$

$$G(x) = \left[\nabla_x \varphi_1(x) \;\vdots\; \nabla_x \varphi_2(x) \;\vdots\; \cdots \;\vdots\; \nabla_x \varphi_L(x)\right]$$

$$\nabla_x F(x) + G(x)\lambda = 0 \qquad G(x)\lambda = -\nabla_x F(x)$$

Unambiguous solution exists if and only if $\operatorname{rank} G(x) = \operatorname{rank}\left[G(x) \;\vdots\; -\nabla_x F(x)\right]$,

which is always true as long as F is convex and φ_l are linear.

How to find irregular solutions?

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Optimization under equality constraints

Lagrange' a multipliers method

If F(x) is continuous, differentiable and convex function and constraints $\varphi_1(x), \varphi_2(x), \ldots, \varphi_L(x)$ are linear then system of equations:

$$\nabla_x L(x,\lambda)\Big|_{x^*,\lambda^*} = 0_S$$

$$\nabla_\lambda L(x,\lambda)\Big|_{x^*,\lambda^*} = 0_L$$

has one solution and it si solution of optimization task.

The above system of equations is necessary and sufficient condition for optimal solution

Optimization under equality constraints

- The generalized method of Lagrange multipliers

Generalized Lagrange function:

$$L(x,\lambda,\lambda_0) = \lambda_0 F(x) + \sum_{l=1}^{L} \lambda_l \varphi_l(x)$$

Necessary conditions of optimality:

$$\nabla_x L(x,\lambda,\lambda_0)\Big|_{x^*,\lambda^*,\lambda_0} = 0_S$$

$$\nabla_\lambda L(x,\lambda,\lambda_0)\Big|_{x^*,\lambda^*,\lambda_0} = 0_L$$

Optimization under equality constraints

- The generalized method of Lagrange multipliers

$$\nabla_x L(x,\lambda,\lambda_0) = \lambda_0 \nabla_x F(x) + \sum_{l=1}^{L} \lambda_l \nabla_x \varphi_l(x) = 0_S$$

1° $\quad \lambda_0 \neq 0 \quad \nabla_x F(x) + \sum_{l=1}^{L} \frac{\lambda_l}{\lambda_0} \nabla_x \varphi_l(x) = 0_S \Rightarrow \nabla_x F(x) + \sum_{l=1}^{L} \lambda_l' \nabla_x \varphi_l(x) = 0_S$

$\lambda_0 = 1 \quad \nabla_x F(x) + \sum_{l=1}^{L} \lambda_l \nabla_x \varphi_l(x) = 0_S$ We obtain regular solutions.

2° $\quad \lambda_0 = 0 \quad \sum_{l=1}^{L} \lambda_l \nabla_x \varphi_l(x) = 0_S$ We obtain irregular solutions.

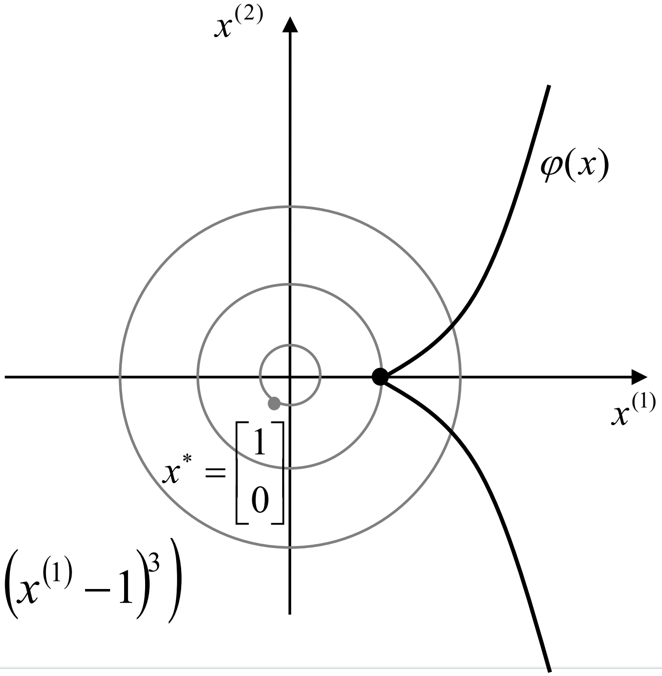
Second order condition of optimality requires analysis of $H(x,\lambda,\lambda_0) = \nabla_{xx}^2 L(x,\lambda,\lambda_0)$.

Optimization under equality constraints

- The generalized method of Lagrange multipliers – – example 2 once again

$$F(x)=\left(x^{(1)}\right)^2+\left(x^{(2)}\right)^2$$

$$\varphi(x)=\left(x^{(2)}\right)^2-\left(x^{(1)}-1\right)^3=0$$



$$L(x,\lambda,\lambda_0)=\lambda_0\left(\left(x^{(1)}\right)^2+\left(x^{(2)}\right)^2\right)+\lambda\left(\left(x^{(2)}\right)^2-\left(x^{(1)}-1\right)^3\right)$$

Example 2.2.2

- $L(x,\lambda) = \lambda_0\left(\left(x^{(1)}\right)^2 + \left(x^{(2)}\right)^2\right) + \lambda\left(\left(x^{(2)}\right)^2 - \left(x^{(1)} - 1\right)^3\right)$

- $$\nabla_x L(x,\lambda) = \begin{bmatrix} \frac{\partial L}{\partial x^{(1)}} \\ \frac{\partial L}{\partial x^{(2)}} \end{bmatrix} = \begin{bmatrix} 2\lambda_0 x^{(1)} - 3\lambda\left(x^{(1)} - 1\right)^2 \\ 2\lambda_0 x^{(2)} + 2\lambda x^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

- $$\nabla_\lambda L(x,\lambda) = \frac{\partial L}{\partial \lambda} = \left(x^{(2)}\right)^2 - \left(x^{(1)} - 1\right)^3 = 0 \quad (3)$$

- Dla $\lambda_0=1$

  For

- $$\nabla_x L(x,\lambda) = \begin{bmatrix} \frac{\partial L}{\partial x^{(1)}} \\ \frac{\partial L}{\partial x^{(2)}} \end{bmatrix} = \begin{bmatrix} 2x^{(1)} - 3\lambda\left(x^{(1)} - 1\right)^2 \\ 2x^{(2)} + 2\lambda x^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

  As before contradiction

- Jak poprzednio - sprzeczność

Example 2.2.2

- $L(x,\lambda) = \lambda_0\left(\left(x^{(1)}\right)^2 + \left(x^{(2)}\right)^2\right) + \lambda\left(\left(x^{(2)}\right)^2 - \left(x^{(1)} - 1\right)^3\right)$
- $$\nabla_x L(x,\lambda) = \begin{bmatrix} \dfrac{\partial L}{\partial x^{(1)}} \\ \dfrac{\partial L}{\partial x^{(2)}} \end{bmatrix} = \begin{bmatrix} 2\lambda_0 x^{(1)} - 3\lambda\left(x^{(1)} - 1\right)^2 \\ 2\lambda_0 x^{(2)} + 2\lambda x^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$
- $$\nabla_\lambda L(x,\lambda) = \frac{\partial L}{\partial \lambda} = \left(x^{(2)}\right)^2 - \left(x^{(1)} - 1\right)^3 = 0 \qquad (3)$$
- Dla $\lambda_0 = 0$

  For
- $$\nabla_x L(x,\lambda) = \begin{bmatrix} \dfrac{\partial L}{\partial x^{(1)}} \\ \dfrac{\partial L}{\partial x^{(2)}} \end{bmatrix} = \begin{bmatrix} 3\lambda\left(x^{(1)} - 1\right)^2 \\ \lambda x^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$
- z (1) $\rightarrow 3\lambda\left(x^{(1)} - 1\right)^2 = 0$ czyli $x^{(1)} = 1$, z (2) $\rightarrow x^{(2)} = 0$

Example 2.2.2

- $\nabla_x L(x,\lambda) = \begin{bmatrix} \frac{\partial L}{\partial x^{(1)}} \\ \frac{\partial L}{\partial x^{(2)}} \end{bmatrix} = \begin{bmatrix} 2\lambda_0 x^{(1)} - 3\lambda\left(x^{(1)} - 1\right)^2 \\ 2\lambda_0 x^{(2)} + 2\lambda x^{(2)} \end{bmatrix}$

- $H_L = \nabla_{xx} L(x,\lambda) = \begin{bmatrix} \frac{\partial^2 L}{\partial^2 x^{(1)}} & \frac{\partial^2 L}{\partial x^{(1)} \partial x^{(2)}} \\ \frac{\partial^2 L}{\partial x^{(2)} \partial x^{(1)}} & \frac{\partial^2 L}{\partial^2 x^{(2)}} \end{bmatrix} =$

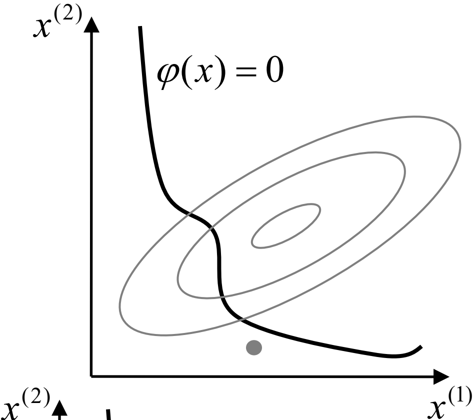
- $\begin{bmatrix} 2\lambda_0 - 6\lambda\left(x^{(1)} - 1\right) & 0 \\ 0 & 2\lambda_0 + 2\lambda \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2\lambda \end{bmatrix}$

Matrix H_L is semi positively defined then point $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ - minimum

- Macierz H_L jest dodatnio pół określona

- Punkt $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ minimum

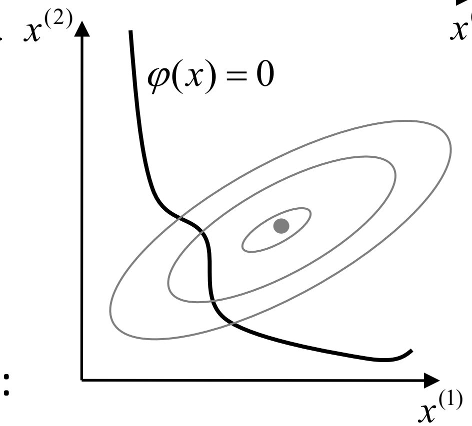
General classification of optimization tasks

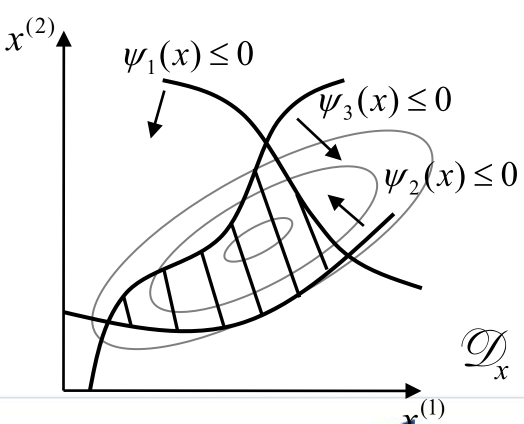


Unconstrained optimization: $\mathscr{D}_x = \mathscr{R}^S$

Optimization under equality constraints:

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \varphi_1(x)=0, \varphi_2(x)=0, \ldots, \varphi_L(x)=0, L \le S \right\}$$





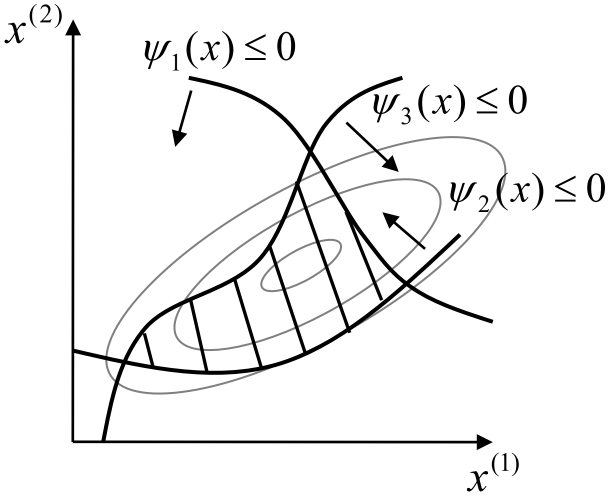
Optimization under inequality constraints:

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \psi_1(x) \le 0, \psi_2(x) \le 0, \ldots, \psi_M(x) \le 0 \right\}$$

Optimization under inequality constraints

Optimization task: $x^* \to F(x^*) = \min_{x^* \in \mathcal{D}_x} F(x)$

$$\mathcal{D}_x = \left\{ x \in \mathcal{R}^S : \psi_1(x) \le 0, \psi_2(x) \le 0, \ldots, \psi_M(x) \le 0 \right\}$$

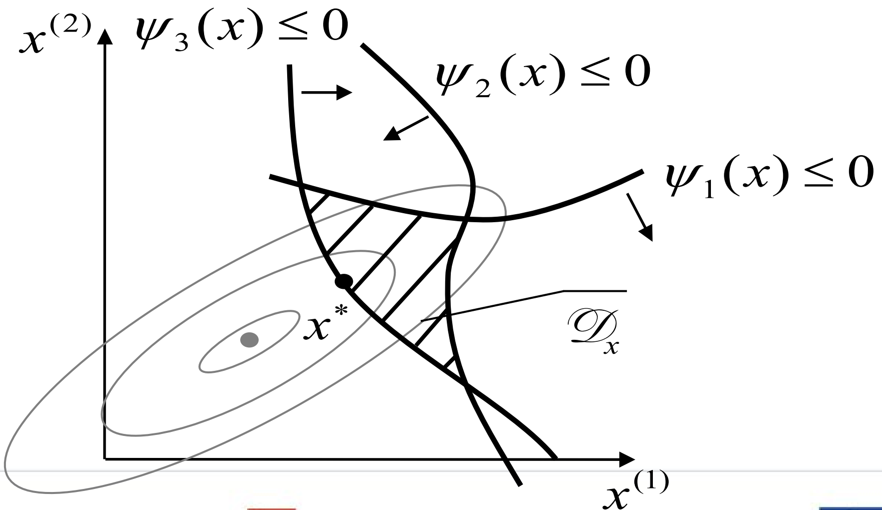


Optimization under inequality constraints

Optimization task

$$x^* \rightarrow F(x^*) = \min_{x \in \mathscr{D}_x} F(x)$$

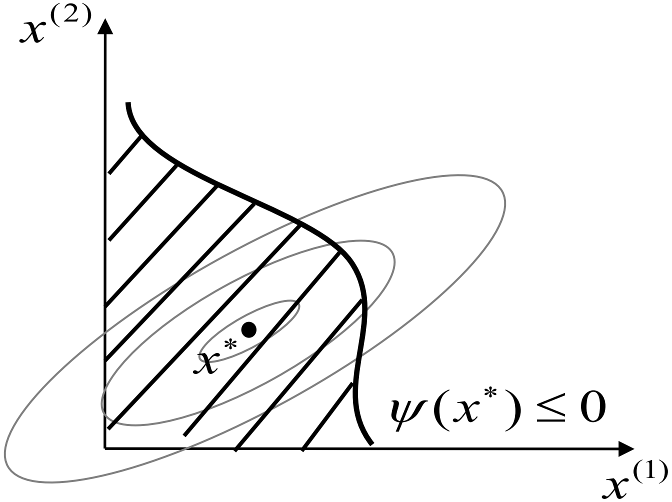
$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \; \psi_1(x) \leq 0, \psi_2(x) \leq 0, \ldots, \psi_M(x) \leq 0, \; \right\}$$



Optimization under inequality constraints

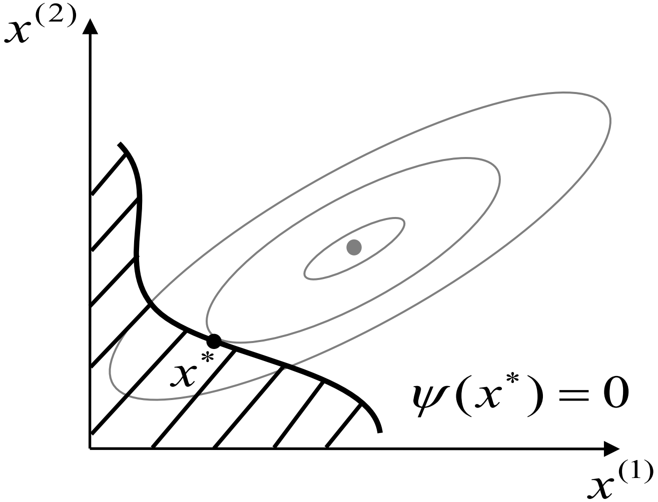
Inactive constraint

$$\psi(x^*) < 0$$



Active constraint

$$\psi(x^*) = 0$$



Optimization under inequality constraints

Kuhn-Tucker conditions

Lagrange function:

$$L(x,\mu) = F(x) + \mu^T \psi(x) \quad \Leftrightarrow \quad L(x,\mu) = F(x) + \sum_{m=1}^{M} \mu_m \psi_m(x)$$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_M \end{bmatrix}$$

Necessary conditions of optimality:

$$\nabla_x L(x,\mu)\Big|_{x^*,\mu^*} = 0_S$$

$$\mu^T \nabla_\mu L(x,\mu)\Big|_{x^*,\mu^*} = 0$$

$$\nabla_\mu L(x,\mu)\Big|_{x^*,\mu^*} \le 0_M$$

$$\mu^* \ge 0_M$$

If solution is regular

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_S \end{bmatrix} \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_S \end{bmatrix}$$

$$\alpha \le \beta \Rightarrow \forall_{s=1,\dots,S} \alpha_s \le \beta_s$$

Optimization under inequality constraints

Kuhn-Tucker conditions

$$\nabla_x L(x,\mu) = \nabla_x F(x) + \sum_{m=1}^{M} \mu_m \nabla_x \psi_m(x) = 0_S$$

$$\mu^T \nabla_\mu L(x,\mu) = \mu^T \psi(x) = \sum_{m=1}^{M} \mu_m \psi_m(x) = 0$$

$$\nabla_\mu L(x,\mu) = \psi(x) \le 0_M$$

$$\mu \ge 0_M$$

$$\mu_1\psi_1(x) + \mu_2\psi_2(x) + \ldots + \mu_M\psi_M(x) = 0$$

$$\forall_m \psi_m(x) \le 0 \qquad \forall_m \mu_m \ge 0$$

$$\forall_m \mu_m \psi_m(x) = 0$$

$$\nabla_x L(x,\mu)\Big|_{x^*,\mu^*} = 0_S$$

$$\mu_m \psi_m(x)\Big|_{x^*,\mu^*} = 0 \qquad m = 1, 2, \ldots, M$$

$$\psi_m(x)\Big|_{x^*,\mu^*} \le 0 \qquad m = 1, 2, \ldots, M$$

$$\mu_m^* \ge 0 \qquad m = 1, 2, \ldots, M$$

Optimization under inequality constraints

Kuhn-Tucker conditions

$$L(x,\mu)=F(x)+\mu_m\psi_m(x)$$

$$\nabla_x L(x,\mu)=\nabla_x F(x)+\mu_m\nabla_x\psi_m(x)=0_S$$

$$\mu^T\nabla_\mu L(x,\mu)=\mu_m\psi_m(x)=0$$

$$\nabla_\mu L(x,\mu)=\psi_m(x)\le 0$$

$$\mu_m\ge 0$$

m - th inactive constraints

$$\mu_m=0 \quad \psi_m(x)<0$$

$$\nabla_x L(x,\mu)=\nabla_x F(x)=0_S$$

$$\nabla_\mu L(x,\mu)=\psi_m(x)<0$$

Like without constraints

m - th active constraints

$$\mu_m>0 \quad \psi_m(x)=0$$

$$\nabla_x L(x,\mu)=\nabla_x F(x)+\mu_m\nabla_x\psi_m(x)=0_S$$

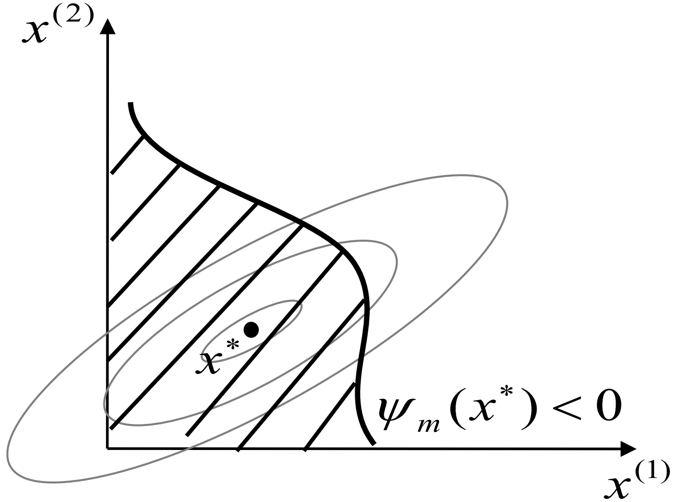
$$\nabla_\mu L(x,\mu)=\psi_m(x)=0$$

Like with equality constraints

Optimization under inequality constraints

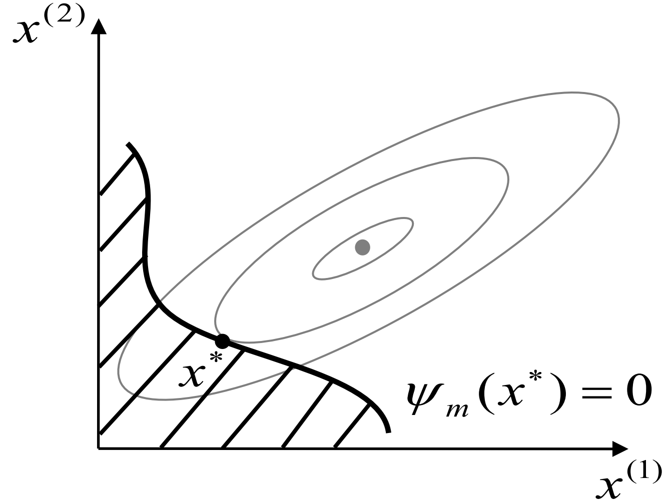
Inactive constraint

$$\psi_m(x^*) < 0$$



Active constraint

$$\psi_m(x^*) = 0$$



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Optimization under inequality constraints

Kuhn-Tucker conditions

$$L(x,\mu) = F(x) + \mu_m \psi_m(x)$$

$$\nabla_x L(x,\mu) = \nabla_x F(x) + \mu_m \nabla_x \psi_m(x) = 0_S$$

$$\mu^T \nabla_\mu L(x,\mu) = \mu_m \psi_m(x) = 0$$

$$\nabla_\mu L(x,\mu) = \psi_m(x) \le 0$$

$$\mu_m \ge 0$$

$\mu_m = 0 \quad \psi_m(x) < 0 \quad m$ -th constraint is inactive

$\mu_m > 0 \quad \psi_m(x) = 0 \quad m$ -th constraint is active

Optimization under inequality constraints

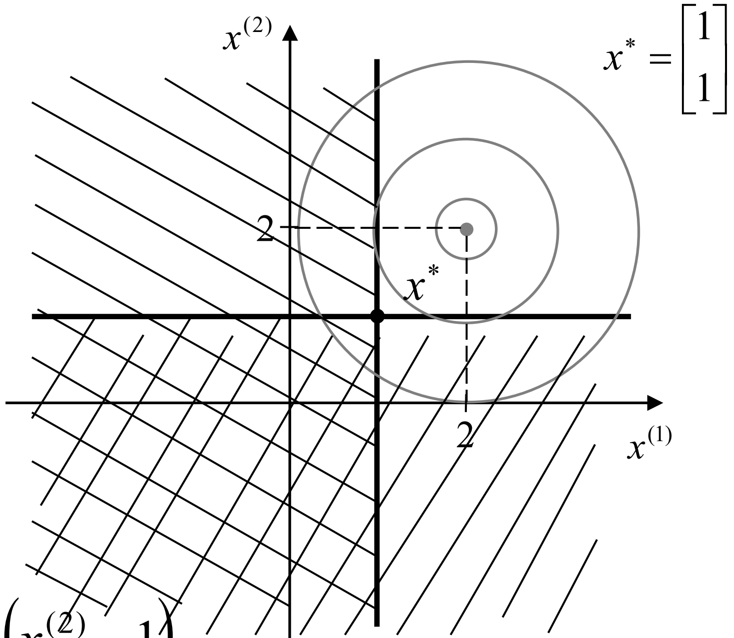
Kuhn-Tucker conditions

Example 1

$$F(x) = \left(x^{(1)} - 2\right)^2 + \left(x^{(2)} - 2\right)^2$$

$$\psi_1(x) = x^{(1)} - 1 \le 0$$

$$\psi_2(x) = x^{(2)} - 1 \le 0$$



$$L(x,\lambda) = \left(x^{(1)} - 2\right)^2 + \left(x^{(2)} - 2\right)^2 + \mu_1\left(x^{(1)} - 1\right) + \mu_2\left(x^{(2)} - 1\right)$$

Example 1.

- $L(x,\mu) = (x^{(1)}-2)^2 + (x^{(2)}-2)^2 + \mu_1(x^{(1)}-1) + \mu_2(x^{(2)}-1)$
- $\nabla_x L(x,\mu) = \begin{bmatrix} 2(x^{(1)}-2)+\mu_1 \\ 2(x^{(2)}-2)+\mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{matrix} (1) \\ (2) \end{matrix}$
- $\mu^T \nabla_\mu L(x,\mu) \sim \left| \begin{matrix} \mu_1(x^{(1)}-1) = 0 \\ \mu_2(x^{(2)}-1) = 0 \end{matrix} \right. \quad \begin{matrix} (3) \\ (4) \end{matrix}$
- $\nabla_\mu L(x,\mu) = \left| \begin{matrix} (x^{(1)}-1) \le 0 \\ (x^{(2)}-1) \le 0 \end{matrix} \right. \quad \begin{matrix} (5) \\ (6) \end{matrix}$
- $\mu = \left| \begin{matrix} \mu_1 \ge 0 \\ \mu_2 \ge 0 \end{matrix} \right. \quad \begin{matrix} (7) \\ (8) \end{matrix}$

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Example 1. c.d.

- 1^0 $\mu_1 = 0\ (x^{(1)} - 1 < 0\ ??), \mu_2 = 0\ (x^{(2)} - 1 < 0\ ??),$

z $(1) \rightarrow\ 2\left(x^{(1)} - 2\right) = 0\ \rightarrow x^{(1)} = 2$

z $(2) \rightarrow\ 2\left(x^{(2)} - 2\right) = 0\ \rightarrow x^{(2)} = 2$

z $(5) \rightarrow (2 - 1) = 1 \geq 0\ sprzczność\ z\ (5)$ contradiction

z $(6) \rightarrow (2 - 1) = 1 \geq 0\ sprzczność\ z\ (6)$ contradiction

- 2^0 $\mu_1 > 0\ (x^{(1)} - 1 = 0\ ??), \mu_2 = 0\ (x^{(2)} - 1 < 0\ ??),$

z $(3) \rightarrow \mu_1\left(x^{(1)} - 1\right) = 0/\mu_1 \rightarrow \left(x^{(1)} - 1\right) = 0 \rightarrow x^{(1)} = 1$

z $(1) \rightarrow 2(1 - 2) + \mu_1 = 0 \rightarrow \mu_1 = 2$

z $(2) \rightarrow\ 2\left(x^{(2)} - 2\right) = 0\ \rightarrow x^{(2)} = 2$

z $(6) \rightarrow (2 - 1) = 1 \geq 0\ sprzczność\ z\ (6)$ contradiction

Example 1. c.d.

- $3^0 \quad \mu_1 = 0 \left(x^{(1)} - 1 < 0\,??\right), \mu_2 > 0 \; (x^{(2)} - 1 = 0\,??),$

z $(1) \rightarrow \; 2\left(x^{(1)} - 2\right) = 0 \; \rightarrow x^{(1)} = 2$

z $(5) \rightarrow (2 - 1) = 1 \geq 0 \; sprzczność \; z \; (5)$ contradiction

z $(4) \rightarrow \mu_2\left(x^{(2)} - 1\right) = 0/\mu_1 \rightarrow \left(x^{(2)} - 1\right) = 0 \rightarrow x^{(2)} = 1$

z $(2) \rightarrow 2(1 - 2) + \mu_2 = 0 \rightarrow \mu_2{=}2$

- $4^0 \quad \mu_1 > 0 \left(x^{(1)} - 1 = 0\,??\right), \mu_2 > 0 \; (x^{(2)} - 1 = 0\,??),$

z $(3) \rightarrow \mu_1\left(x^{(1)} - 1\right) = 0/\mu_1 \rightarrow \left(x^{(1)} - 1\right) = 0 \rightarrow x^{(1)} = 1$

z $(1) \rightarrow 2(1 - 2) + \mu_1 = 0 \rightarrow \mu_1{=}2$

z $(4) \rightarrow \mu_2\left(x^{(2)} - 1\right) = 0/\mu_1 \rightarrow \left(x^{(2)} - 1\right) = 0 \rightarrow x^{(1)} = 1$

z $(2) \rightarrow 2(1 - 2) + \mu_2 = 0 \rightarrow \mu_2{=}2$

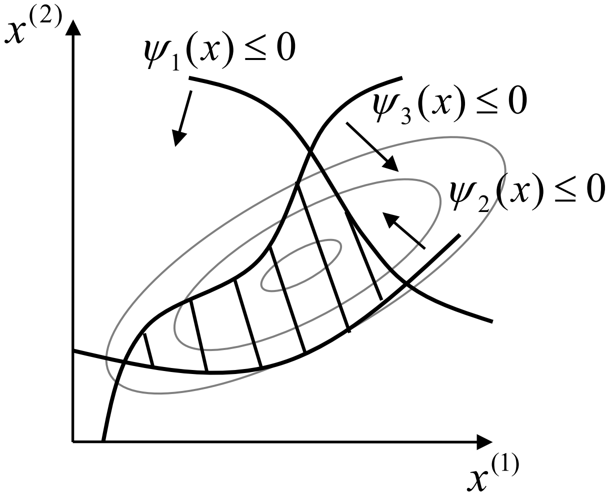
Point $x^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ optimal solution

Punkt $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ spełnia równania i jest rozwiązaniem zadania

Optimization under inequality constraints

Optimization task: $x^* \rightarrow F(x^*) = \min_{x^* \in \mathscr{D}_x} F(x)$

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \psi_1(x) \le 0, \psi_2(x) \le 0, \ldots, \psi_M(x) \le 0 \right\}$$



Optimization under inequality constraints

Kuhn-Tucker conditions

Lagrange' a function :

$$L(x,\mu)=F(x)+\mu^T\psi(x)\ \Leftrightarrow\ L(x,\mu)=F(x)+\sum_{m=1}^{M}\mu_m\psi_m(x)$$

where: $\mu=\begin{bmatrix}\mu_1\\ \mu_2\\ \vdots\\ \mu_M\end{bmatrix}$ - Vector of Lgrange' a multiplayers

Neccesery conditions:

$$\nabla_x L(x,\mu)\big|_{x^*,\mu^*}=0_S$$

$$\mu^T\nabla_\mu L(x,\mu)\big|_{x^*,\mu^*}=0$$

$$\nabla_\mu L(x,\mu)\big|_{x^*,\mu^*}\le 0_M$$

$$\mu^*\ge 0_M$$

$\Leftrightarrow$ The solution is regular solution

$$\alpha=\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_S\end{bmatrix}\quad \beta=\begin{bmatrix}\beta_1\\ \beta_2\\ \vdots\\ \beta_S\end{bmatrix}$$

$$\alpha\le\beta\Rightarrow\forall_{s=1,\ldots,S}\ \alpha_s\le\beta_s$$

Optimization under inequality constraints

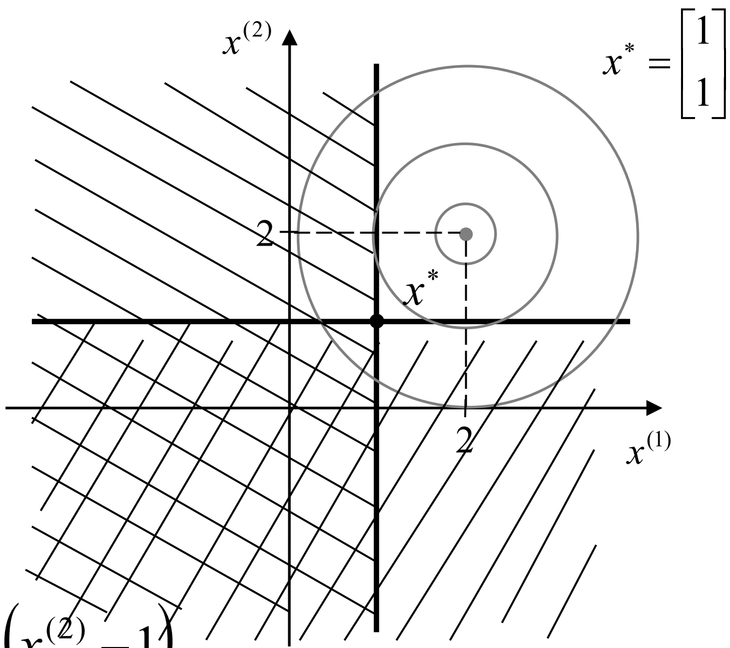
Kuhn-Tucker conditions

Example 1

$$F(x) = \left(x^{(1)} - 2\right)^2 + \left(x^{(2)} - 2\right)^2$$

$$\psi_1(x) = x^{(1)} - 1 \le 0$$

$$\psi_2(x) = x^{(2)} - 1 \le 0$$



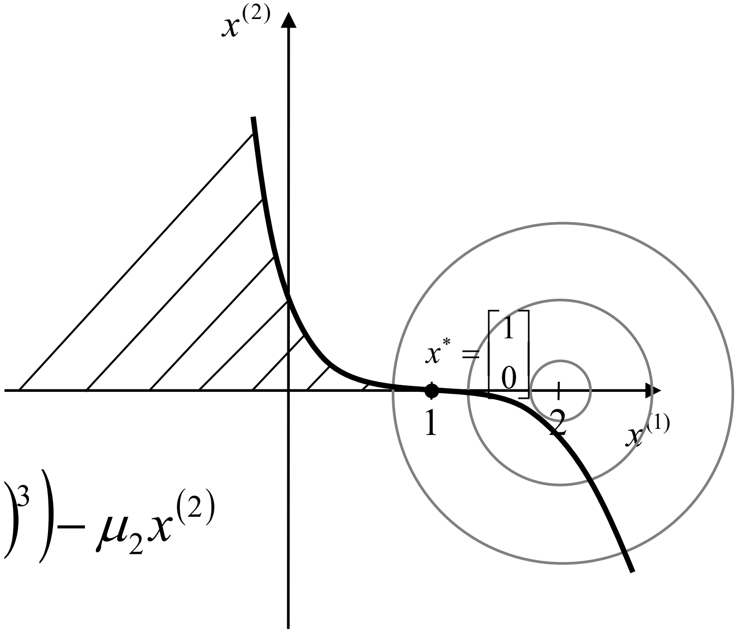
$$L(x,\lambda) = \left(x^{(1)} - 2\right)^2 + \left(x^{(2)} - 2\right)^2 + \mu_1\left(x^{(1)} - 1\right) + \mu_2\left(x^{(2)} - 1\right)$$

Optimization under inequality constraints

Kuhn-Tucker conditions

Example 2 – irregular

$$F(x) = \left(x^{(1)} - 2\right)^2 + \left(x^{(2)}\right)^2$$
$$\psi_1(x) = x^{(2)} + \left(x^{(1)} - 1\right)^3 \le 0$$
$$\psi_2(x) = -x^{(2)} \le 0$$



$$L(x,\lambda) = \left(x^{(1)} - 2\right)^2 + \left(x^{(2)}\right)^2 + \mu_1\left(x^{(2)} + \left(x^{(1)} - 1\right)^3\right) - \mu_2 x^{(2)}$$

Example 2.

- $L(x, \mu) = \left(x^{(1)} - 2\right)^2 + \left(x^{(2)} - 2\right)^2 + \mu_1\left(x^{(2)} + \left(x^{(1)} - 1\right)^3\right) - \mu_2 x^{(2)}$
- $\nabla_x L(x, \mu) = \begin{bmatrix} 2\left(x^{(1)} - 2\right) + 3\mu_1\left(x^{(1)} - 1\right)^2 \\ 2\left(x^{(2)} - 2\right) + \mu_1 - \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (1) (2)
- $\mu^T \nabla_\mu L(x, \mu) \sim \left| \begin{matrix} \mu_1\left(x^{(2)} + \left(x^{(1)} - 1\right)^3\right) = 0 \\ -\mu_2 x^{(2)} = 0 \end{matrix} \right.$ (3) (4)
- $\nabla_\mu L(x, \mu) = \left| \begin{matrix} x^{(2)} + \left(x^{(1)} - 1\right)^3 \leq 0 \\ -x^{(2)} \leq 0 \end{matrix} \right.$ (5) (6)
- $\mu = \left| \begin{matrix} \mu_1 \geq 0 \\ \mu_2 \geq 0 \end{matrix} \right.$ (7) (8)

contradiction

Example 2.

The above system o equation ought to be solved as before. For each cases it can be shown contradiction.

We will show that solution $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which can be notice from graphical illustratio does not fulfill system of equations

Example 2.

For we obtain

- Dla $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ otrzymujemy

contradiction

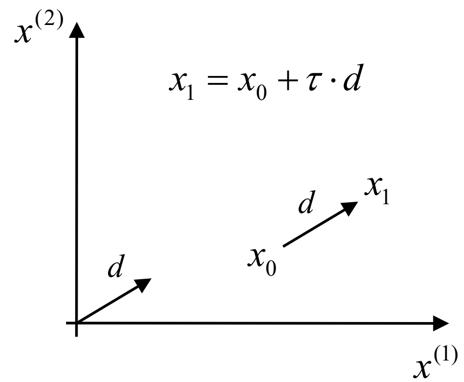
- $\nabla_x L(x, \mu\ ) = \begin{bmatrix} 2(1-2) + 3\mu_1(1-1)^2 = -2 \neq 0 \ \ sprzeczno\acute{c} \\ 2(0-2) + \mu_1 - \mu_2 = 0 \end{bmatrix}$ (1) (2)
- $\mu^T \nabla_\mu L(x, \mu\ ) \sim \begin{vmatrix} \mu_1(0 + (1-1)^3) = 0 \\ -\mu_2 0 = 0 \end{vmatrix}$ (3) (4)
- $\nabla_\mu L(x, \mu\ ) = \begin{vmatrix} 0 + (1-1)^3 \leq 0 \\ -0 \leq 0 \end{vmatrix}$ (5) (6)
- $\mu = \begin{vmatrix} \mu_1 \geq 0 \\ \mu_2 \geq 0 \end{vmatrix}$ (7) (8)

Rozwiązanie nieregularne Irregular solution

Optimization under inequality constraints

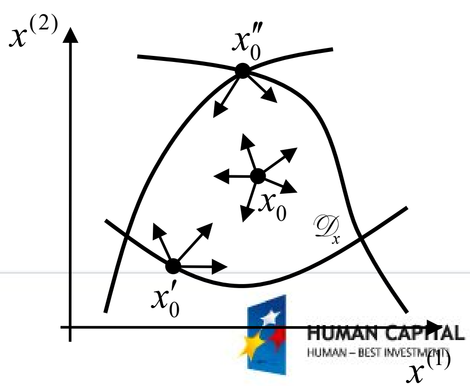
Feasible directions

$$d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_S \end{bmatrix}$$ – direction in $\mathscr{R}^S$



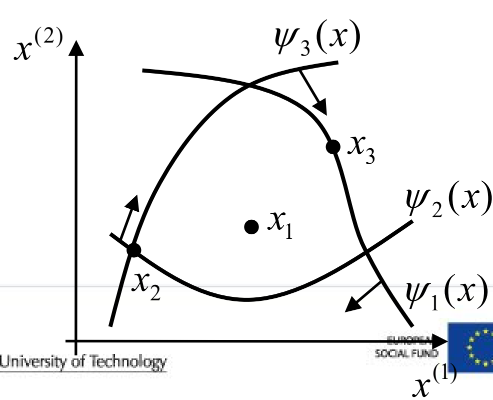
Feasible directions:

$$D(x) = \left\{ d \in \mathscr{R}^s : \exists \tau \quad x + \tau d \in \mathscr{D}_x \right\}$$



Active constraints:

$$I(x) = \left\{ m \in \{1, 2, \ldots, M\} : \psi_m(x) = 0 \right\}$$



$$I(x_1) = \varnothing$$
$$I(x_2) = \{2, 3\}$$
$$I(x_3) = \{1\}$$

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Optimization under inequality constraints

Kuhn – Tucker rolls

Active constraints – analytical conditions?

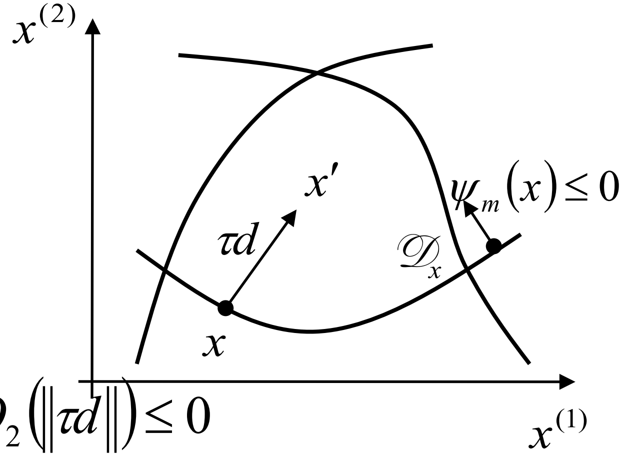
$$D(x) = \{ d \in \mathscr{R}^s : \exists \tau \quad x + \tau d \in \mathscr{D}_x \}$$

$$\forall m \in I(x)$$

tj.: $\psi_m(x) = 0$

$$x' = x + \tau d \in \mathscr{D}_x, \ \ \tau > 0$$

$$\psi_m(x') \le 0$$



$$\psi_m(x') = \psi_m(x + \tau d) = \psi_m(x) + \tau d^T \nabla_x \psi_m(x) + O_2(\|\tau d\|) \le 0$$

$$\tau d^T \nabla_x \psi_m(x) \le 0 \quad \tau > 0$$

$$d^T \nabla_x \psi_m(x) \le 0$$ – analytical condition

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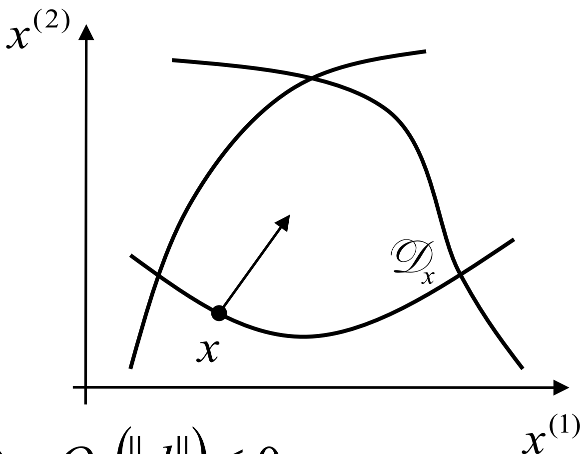
Optimization under inequality constraints

Feasible directions

$$\forall d \in D(x) \wedge \forall m \in I(x)$$

How to determine the set of feasible directions?



Active constraints $\psi_m(x) = 0$

$$x' = x + \tau d \in \mathscr{D}_x$$

$$\psi_m(x') \le 0$$

$$\psi_m(x') = \psi_m(x + \tau d) = \psi_m(x) + \tau d^T \nabla_x \psi_m(x) + O_2(\|d\|) \le 0$$

$$\tau d^T \nabla_x \psi_m(x) \le 0$$

$$\forall d \in D(x) \wedge \forall m \in \mathrm{I}(x) \implies d^T \nabla_x \psi_m(x) \le 0$$ – analytical condition

$$\mathscr{D}(x) = \left\{ d \in \mathscr{R}^s : \forall_m \in I(x),\ d^T \nabla_x \psi_m(x) \le 0 \right\} \qquad \mathscr{D}(x) \ne D(x)$$

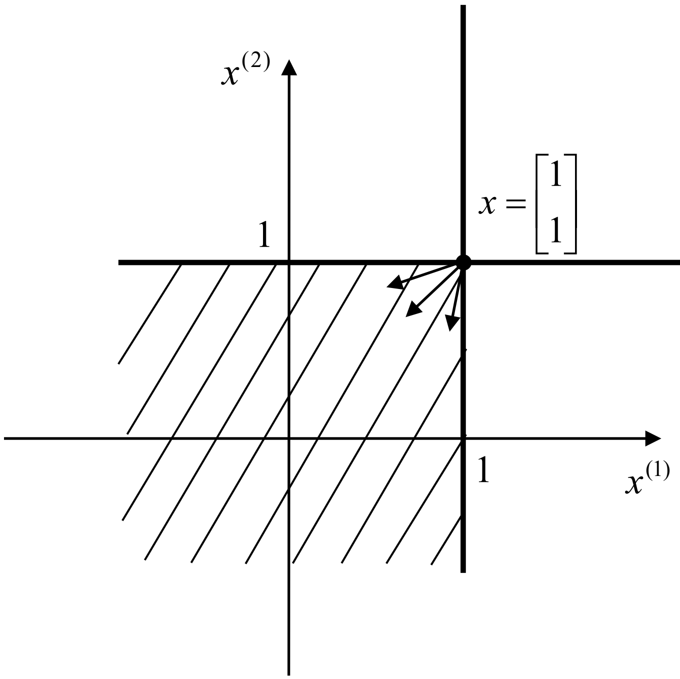
Optimization under inequality constraints

Feasible directions

Example

$$x^{(1)} - 1 \le 0$$

$$x^{(2)} - 1 \le 0$$



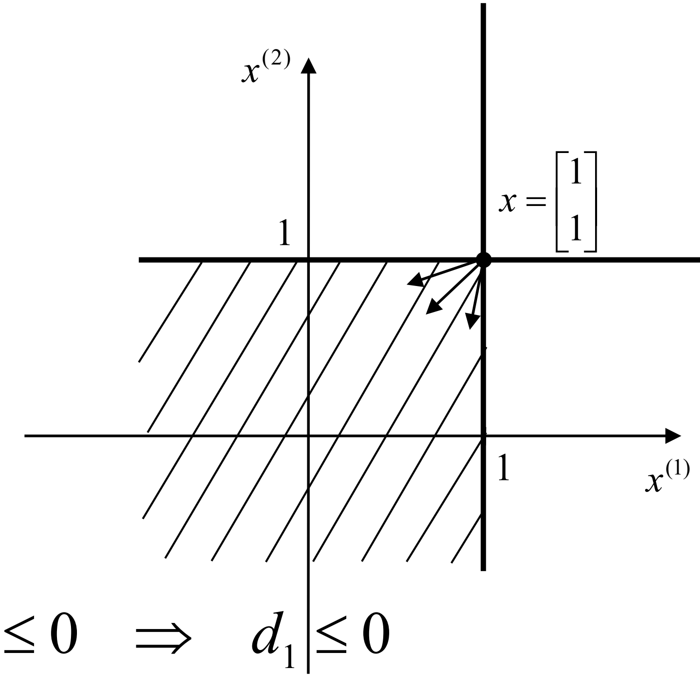
Optimization under inequality constraints

Kuhn – Tucker rolls

Example 1

$$\psi_1(x) = x^{(1)} - 1 \le 0$$

$$\psi_2(x) = x^{(2)} - 1 \le 0$$



In the point $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad I\left(x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \{1, 2\}$

$$\nabla_x \psi_1(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \nabla_x \psi_2(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$d^T \nabla_x \psi_1(x) = \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \cdot d_1 + 0 \cdot d_2 \le 0 \quad \Rightarrow \quad d_1 \le 0$$

$$d^T \nabla_x \psi_2(x) = \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \cdot d_1 + 1 \cdot d_2 \le 0 \quad \Rightarrow \quad d_2 \le 0$$

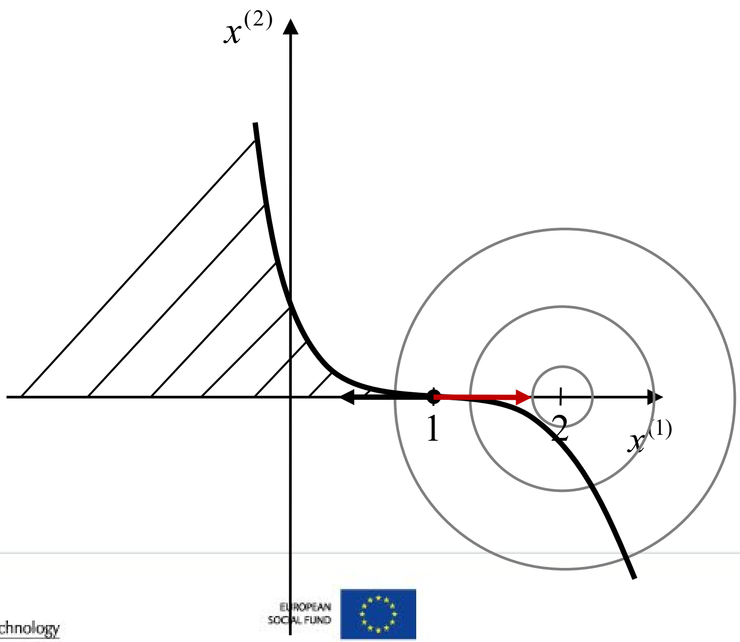
Optimization under inequality constraints

Feasible directions

$$\mathscr{D}(x) = \left\{ d \in \mathscr{R}^s : \forall_m \in I(x),\ d^T \nabla_x \psi_m(x) \le 0 \right\}$$

$\mathscr{D}(x) \neq D(x)$ leads to irregular case

$$F(x) = \left(x^{(1)} - 2\right)^2 + \left(x^{(2)}\right)^2$$
$$\psi_1(x) = x^{(2)} - \left(x^{(1)} - 1\right)^3 \le 0$$
$$\psi_2(x) = -x^{(2)} \le 0$$



Optimization under inequality constraints

Kuhn – Tucker conditions

Attention: Not all direction, which fulfils condition $d^T \nabla_x \psi_m(x) \le 0$ is feasible direction. It mays generate irregular solution

$$\mathscr{D}(x) = \left\{ d \in \mathscr{R}^s : \forall_m \in I(x),\ d^T \nabla_x \psi_m(x) \le 0 \right\} \qquad \mathscr{D}(x) \ne D(x)$$

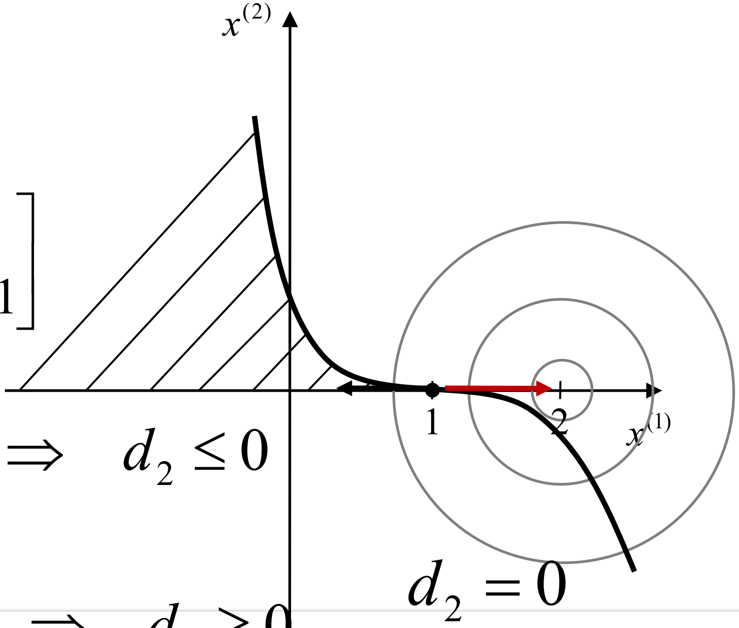
$$\psi_1(x) = x^{(2)} + \left(x^{(1)} - 1\right)^3 \le 0 \qquad \psi_2(x) = -x^{(2)} \le 0 \qquad F(x) = \left(x^{(1)} - 2\right)^2 + \left(x^{(2)}\right)^2$$

In the point $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad I\left(x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \{1, 2\}$

$$\nabla_x \psi_1(x) = \begin{bmatrix} 3\left(x^{(1)} - 1\right)^2 \\ 1 \end{bmatrix}_{x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \nabla_x \psi_2(x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$d^T \nabla_x \psi_1(x) = \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \cdot d_1 + 1 \cdot d_2 \le 0 \quad \Rightarrow \quad d_2 \le 0$$

$$d^T \nabla_x \psi_2(x) = \begin{bmatrix} d_1 & d_2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = 0 \cdot d_1 - 1 \cdot d_2 \le 0 \quad \Rightarrow \quad d_2 \ge 0$$



$d_2 = 0$

$d_1 - any$

Optimization under inequality constraints
Kuhn – Tucker conditions

$$F(x) = F(x_0 + \tau d) = F(x_0) + \tau \left(\nabla_x F(x)\right)^T d + O_3\left(\|\tau d\|\right)$$

If d such that: $\left(\nabla_x F(x)\right)^T d < 0 \quad to \quad F(x) < F(x_0)$ down ↘

Then

Let us divide set of directions $\mathscr{D}(x) = \left\{ d \in \mathscr{R}^s : \forall_m \in I(x),\ d^T \nabla_x \psi_m(x) \le 0 \right\}$:

$$\mathscr{D}_1(x) = \left\{ d \in \mathscr{R}^s : \forall_m \in I(x),\ d^T \nabla_x \psi_m(x) \le 0 \right\} \wedge \left(\nabla_x F(x)\right)^T d \ge 0$$ up ↗

$$\mathscr{D}_2(x) = \left\{ d \in \mathscr{R}^s : \forall_m \in I(x),\ d^T \nabla_x \psi_m(x) \le 0 \right\} \wedge \left(\nabla_x F(x)\right)^T d < 0$$ down ↘

Optimization under inequality constraints

Lagrange function:

Kuhn – Tucker conditions

$$L(x,\mu)=F(x)+\mu^T\psi(x) \Rightarrow L(x,\mu)=F(x)+\sum_{m=1}^{M}\mu_m\psi_m(x)$$

Kuhna – Tucker theorem – necessary optimality conditions:

If x^* is local minimum of optimization problem with inequality constraints, functions : $F, \Psi_1, \Psi_2, \ldots, \Psi_M$ are continuous and function F is differentiable then there exists set of Lagrange μ^* such one that together with x^* fulfils

$$\nabla_x L(x,\mu)\Big|_{x^*,\mu^*}=0_S$$

$$\mu^T\nabla_\mu L(x,\mu)\Big|_{x^*,\mu^*}=0$$

$$\nabla_\mu L(x,\mu)\Big|_{x^*,\mu^*}\le 0_M$$

$$\mu^*\ge 0_M$$

Regular solution

$$\Leftrightarrow \quad \mathscr{D}_2(x)=\left\{d\in\mathscr{R}^s:\forall_m\in I(x),\ d^T\nabla_x\psi_m(x)\le 0\right\}\wedge\left(\nabla_x F(x)\right)^T d<0=\Phi$$

Optimization under inequality constraints Kuhn – Tucker rolls

Regularity Conditions

1. Karlin: constraints $\psi_1(x), \psi_2(x), \ldots, \psi_M(x)$ - linear
2. Slater: constraints $\psi_1(x), \psi_2(x), \ldots, \psi_M(x)$ - convex functions and feasible set is not empty

3.Fiacco – Mac Cormica: in the optimal point gradients of all active constraints are linear independent, i.e.:

$$\forall\, m \in I(x^*) \quad \nabla_x \psi_m(x^*)\Big|_{x=x^*}$$ are linear independent

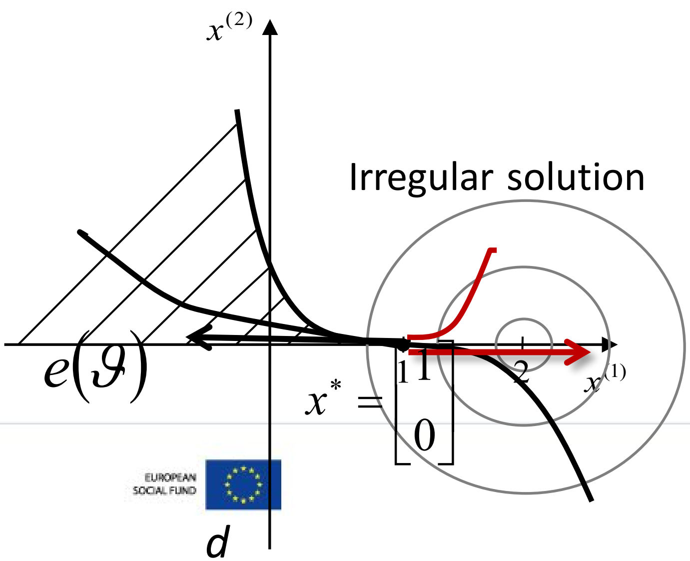
4. Zangwil: $\mathscr{D}(x^*) = \overline{D}(x^*)$

5. Kuhna – Tucker'a: for each direction $d \in \mathscr{D}(x^*)$ there exists regular curve starting in the point x^* tangent to that direction

$$\forall\, d \in \mathscr{D}(x^*) \quad \exists\, e(\vartheta), \quad \vartheta \in [0, 1]$$

$$e(\vartheta) = \begin{bmatrix} e_1(\vartheta) \\ e_2(\vartheta) \\ \vdots \\ e_S(\vartheta) \end{bmatrix}$$

- $e(0) = x^*$
- $e(\vartheta) \in D_x \quad \forall\, \vartheta \in [0, 1]$
- $\left.\dfrac{de(\vartheta)}{d\vartheta}\right|_{\vartheta=0} = \tau \cdot d$



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Irregular solution - Fiacco – Mac Cormica roll

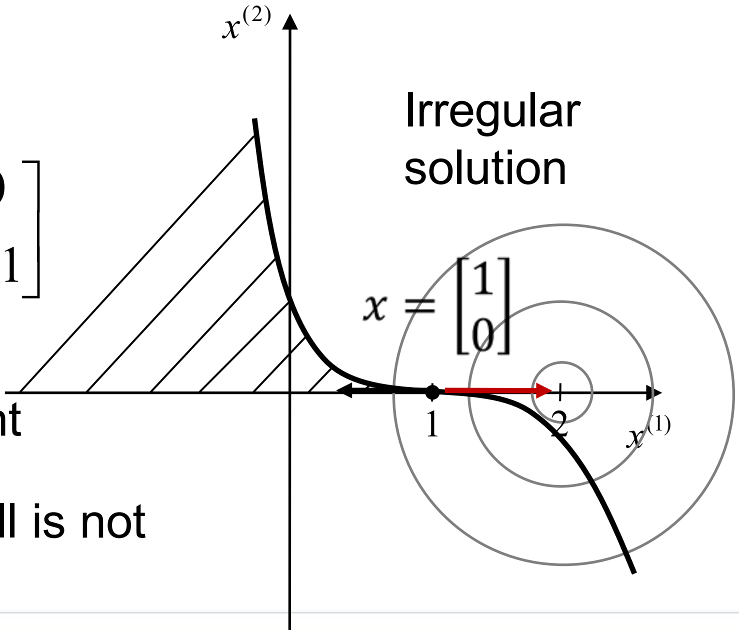
$$\psi_1(x) = x^{(2)} + \left(x^{(1)} - 1\right)^3 \le 0 \qquad \psi_2(x) = -x^{(2)} \le 0 \qquad F(x) = \left(x^{(1)} - 2\right)^2 + \left(x^{(2)}\right)^2$$

In the point $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad I\left(x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \{1, 2\}$

Consrtaints1 i 2 are active

$$\nabla_x \psi_1(x) = \begin{bmatrix} 3\left(x^{(1)} - 1\right)^2 \\ 1 \end{bmatrix}\Bigg|_{x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \nabla_x \psi_2(x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Gradients of constraints are linearly dependent



In the point $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ Fiacco – Mac Cormica roll is not fulfilled

Optimization under inequality constraints

Kuhn-Tucker conditions

Sufficient condition of regularity:

$F, \psi_1, \psi_2, \ldots, \psi_M$ – continuous and differentiable

F – pseudo-convex

$\psi_1, \psi_2, \ldots, \psi_M$ – quasi-convex

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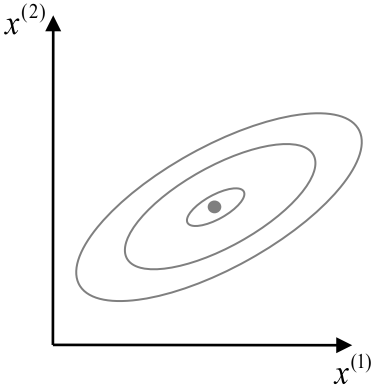
General classification of optimization tasks

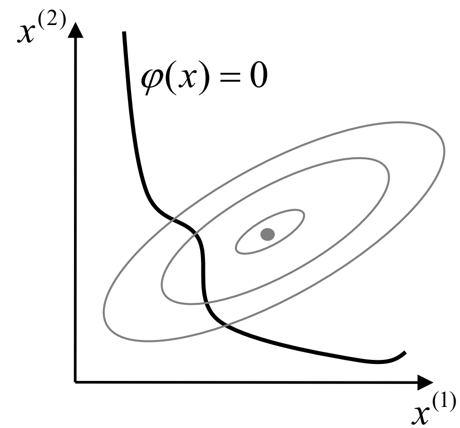


Unconstrained optimization:

$$\mathscr{D}_x = \mathscr{R}^S$$

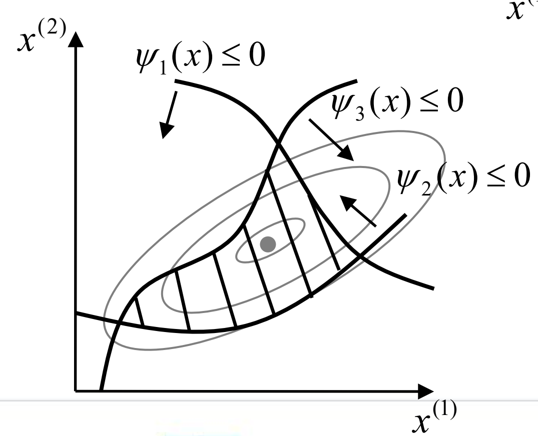
Optimization under equality constraints:

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \varphi_1(x) = 0, \varphi_2(x) = 0, \ldots, \varphi_L(x) = 0, L \le S \right\}$$



Optimization under inequality constraints:

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \psi_1(x) \le 0, \psi_2(x) \le 0, \ldots, \psi_M(x) \le 0 \right\}$$



Optimization under inequality constraints

Kuhn – Tucker rolls

Necessary and sufficient conditions :

If functions $F(x), \psi_1(x), \psi_2(x), \ldots, \psi_M(x)$ are continuous and differentiable and function $F(x)$ is pseudo – convex function , and constraints $\psi_1(x), \psi_2(x), \ldots, \psi_M(x)$ are quasi – convex function then system of equations :

$$\nabla_x L(x,\mu)\Big|_{x^*,\mu^*} = 0_S$$

$$\mu^T \nabla_\mu L(x,\mu)\Big|_{x^*,\mu^*} = 0$$

$$\nabla_\mu L(x,\mu)\Big|_{x^*,\mu^*} \leq 0_M$$

$$\mu^* \geq 0_M$$

Has one solution and it is the solution of the optimisation task with inequality constraints

Optimization under equality constraints

- The method of Lagrange multipliers

Lagrange function:

$$L(x,\lambda) = F(x) + \sum_{l=1}^{L} \lambda_l \varphi_l(x) = F(x) + \lambda^T \varphi(x)$$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_L \end{bmatrix}, \quad \varphi(x) = \begin{bmatrix} \varphi_1(x) \\ \varphi_2(x) \\ \vdots \\ \varphi_L(x) \end{bmatrix}$$

Necessary conditions of optimality:

$$\nabla_x L(x,\lambda)\big|_{x^*,\lambda^*} = 0_S$$

$$\nabla_\lambda L(x,\lambda)\big|_{x^*,\lambda^*} = 0_L$$

If and only if $\operatorname{rank} G(x) = \operatorname{rank}\left[G(x) \;\vdots\; -\nabla_x F(x)\right]$,

Where: $G(x) = \left[\nabla_x \varphi_1(x) \;\vdots\; \nabla_x \varphi_2(x) \;\vdots\; \cdots \;\vdots\; \nabla_x \varphi_L(x)\right]$

Optimization under inequality constraints

Lagrange function: Kuhn – Tucker conditions

$$L(x,\mu) = F(x) + \mu^T \psi(x) \Rightarrow L(x,\mu) = F(x) + \sum_{m=1}^{M} \mu_m \psi_m(x)$$

Kuhna – Tucker theorem – necessary optimality conditions:

If x^* is local minimum of optimization problem with inequality constraints, functions $:F, \Psi_1, \Psi_2, \ldots, \Psi_M$ are continuous and function F is differentiable then there exists set of Lagrange μ^* such one that together with x^* fulfils

$$\nabla_x L(x,\mu)\Big|_{x^*,\mu^*} = 0_S$$

$$\mu^T \nabla_\mu L(x,\mu)\Big|_{x^*,\mu^*} = 0$$

$$\nabla_\mu L(x,\mu)\Big|_{x^*,\mu^*} \le 0_M$$

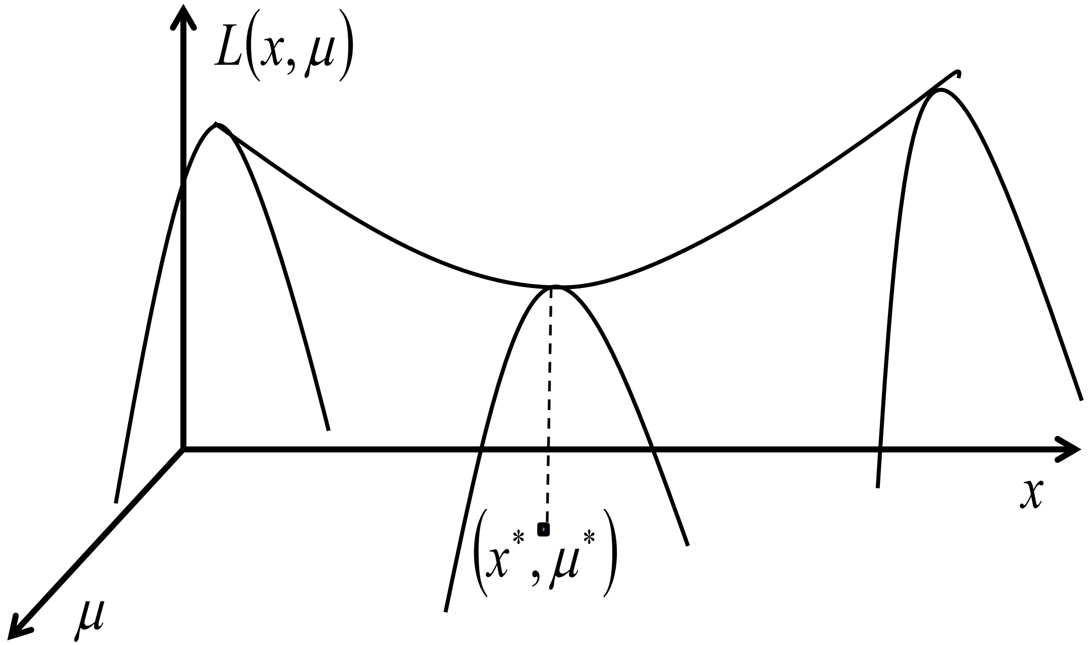
$$\mu^* \ge 0_M$$

Regular solution

$$\Leftrightarrow \quad \mathcal{D}_2(x) = \left\{ d \in \mathcal{R}^s : \forall_m \in I(x),\ d^T \nabla_x \psi_m(x) \le 0 \right\} \wedge \left(\nabla_x F(x)\right)^T d < 0 = \Phi$$

Saddle point

Saddle point (x^*, μ^*)



$$L(x^*, \mu^*) \leq L(x, \mu^*) \quad \forall x \in \mathscr{D}(x) \subseteq \mathscr{R}^S$$

$$L(x^*, \mu) \leq L(x^*, \mu^*) \quad \forall \mu \geq 0_M$$

$$L(x^*, \mu^*) = \min_{x \in \mathscr{D}(x)} \max_{\mu \geq 0_M} L(x, \mu)$$

Saddle point

Point $\left(x^*, \mu^*\right)$ is the saddle poin $\left(x^* \in \mathscr{D}(x),\ \mu \geq 0_M\right) \quad \Leftrightarrow$

$$1.\quad x^* - \min imizing\ L(x,\mu)$$

$$2.\quad \psi_m\left(x^*\right) \leq 0 \quad m = 1, 2, \ldots, M$$

$$3.\quad \mu^* \psi_m\left(x^*\right) = 0 \quad m = 1, 2, \ldots, M$$

If $\left(x^*, \mu^*\right)$ is the saddle point Lagrange'a function *L(x,μ)* then $\left(x^*, \mu^*\right)$ is the solution of the optimization task:

$$x^* \to F(x^*) = \min_{x^* \in \mathscr{D}_x} F(x)$$

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \psi_1(x) \leq 0, \psi_2(x) \leq 0, \ldots, \psi_M(x) \leq 0 \right\}$$

Special case

$$x^* \to F(x^*) = \min_{x^* \in \mathscr{D}_x} F(x)$$

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : x \geq 0_S, \psi(x) \leq 0_M \right\}$$

$$L(x,\mu) = F(x) + \mu^T \psi(x)$$

$$\nabla_x L(x,\mu)\Big|_{x^*,\mu^*} \geq 0_S$$

$$x^T \nabla_x L(x,\mu)\Big|_{x^*,\mu^*} = 0$$

$$x^* \geq 0_S$$

$$\nabla_\mu L(x,\mu)\Big|_{x^*,\mu^*} \leq 0_S$$

$$\mu^T \nabla_\mu L(x,\mu)\Big|_{x^*,\mu^*} = 0$$

$$\mu^* \geq 0_M$$

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : x \geq 0_S, \psi(x) \leq 0_M \right\}$$

$$L(x, \mu) = F(x) + \mu^T \psi(x)$$

$$\mathscr{D}_x = \{ x \in \mathscr{R}^S : \psi(x) \leq 0_M, -x \leq 0_S \}$$

$$L(x. \boxed{?}, \mu') = F(x) + \mu^T \psi(x) - \mu'^T x$$

Kuhn-Tucker conditions

$$L(x,\mu,\mu') = \mathrm{F}(x) + \mu^T \psi(x) - \mu'^T x$$

$$\nabla_x L(x,\mu,\mu') = \mathrm{F}(x) + \sum_{m=1}^{M} \mu_m \nabla_x \psi_m(x) - \mu' = 0_S$$

$$\mu^T \nabla_x L(x,\mu,\mu') = \mu^T \psi(x) = 0$$

$$\nabla_\mu L(x,\mu,\mu') = \psi(x) \le 0_M$$

$$\nabla_{\mu'} L(x,\mu,\mu') = -x \le 0_S$$

$$\mu \ge 0_M,\ \mu' \ge 0_S$$

Special case

$$x^* \to F(x^*) = \min_{x^* \in \mathscr{D}_x} F(x)$$

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : x \geq 0_S, \psi(x) \leq 0_M \right\}$$

$$L(x, \mu) = F(x) + \mu^T \psi(x)$$

$$\nabla_x L(x, \mu)\Big|_{x^*, \mu^*} \geq 0_S$$

$$x^T \nabla_x L(x, \mu)\Big|_{x^*, \mu^*} = 0$$

$$x^* \geq 0_S$$

$$\nabla_\mu L(x, \mu)\Big|_{x^*, \mu^*} \leq 0_S$$

$$\mu^T \nabla_\mu L(x, \mu)\Big|_{x^*, \mu^*} = 0$$

$$\mu^* \geq 0_M$$

Special case

$$x^* \to F(x^*) = \min_{x^* \in \mathcal{D}_x} F(x)$$

$$\mathcal{D}_x = \left\{ x \in \mathcal{R}^S : \varphi(x) = 0_L ,\ \psi(x) \le 0_M \right\}$$

$$L(x,\lambda,\mu) = F(x) + \lambda^T \varphi(x) + \mu^T \psi(x)$$

$$\nabla_x L(x,\lambda,\mu)\Big|_{x^*,\lambda^*,\mu^*} = 0_S$$

$$\nabla_\lambda L(x,\lambda,\mu)\Big|_{x^*,\lambda^*,\mu^*} = 0_L$$

$$\mu^T \nabla_\mu L(x,\lambda,\mu)\Big|_{x^*,\lambda^*,\mu^*} = 0$$

$$\nabla_\mu L(x,\lambda,\mu)\Big|_{x^*,\lambda^*,\mu^*} \le 0_M$$

$$\mu^* \ge 0_S$$

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \varphi(x) = 0_L, \psi(x) \le 0_M \right\}$$

$$L(x, \lambda, \mu) = F(x) + \lambda^T \varphi(x) + \mu^T \psi(x)$$

$$\varphi(x) = 0_L \equiv \varphi(x) \le 0_L \cap -\varphi(x) \le 0_L$$

$$\mathscr{D}_x = \{ x \in \mathscr{R}^S : \varphi(x) \le 0_L \cap -\varphi(x) \le 0_L, \psi(x) \le 0_M \}$$

$$L(x, \lambda', \lambda, \mu) = F(x) + \lambda^T \varphi(x) - \lambda'^T \varphi(x) + \mu^T \psi(x)$$

Warunki Kuhna-Tuckera

$$L(x,\lambda,\lambda',\mu) = F(x)+\lambda^T\varphi(x)-\lambda'^T\varphi(x)+\mu^T\psi(x)$$

$$\nabla_x L(x,\lambda,\lambda',\mu) =$$

$$= \nabla_x F(x)+\sum_{l=1}^{L}\lambda_l\nabla_x\varphi_l(x) - \sum_{l=1}^{L}\lambda'_l\nabla_x\varphi_l(x) + \sum_{m=1}^{M}\mu_m\nabla_x\psi_m(x) = 0_S$$

$$\lambda^T\nabla_\lambda L(x,\lambda,\lambda',\mu) = \lambda^T\varphi(x) = 0$$

$$\lambda'^T\nabla_{\lambda'} L(x,\lambda,\lambda',\mu) = -\lambda'^T\varphi(x) = 0$$

$$\mu^T\nabla_\mu L(x,\lambda,\lambda',\mu) = \mu^T\psi(x) = 0$$

$$\nabla_\lambda L(x,\lambda,\lambda',\mu) = \varphi(x) \leq 0_L$$

$$\nabla_{\lambda'} L(x,\lambda,\lambda',\mu) = -\varphi(x) \leq 0_L$$

$$\nabla_\mu L(x,\lambda,\lambda',\mu) = \psi(x) \leq 0_M$$

$$\lambda \geq 0_L, \lambda' \geq 0_L, \mu \geq 0_M$$

$$\mathscr{D}_x = \left\{ x \in \mathscr{R}^S : \varphi(x) = 0_L, \psi(x) \le 0_M \right\}$$

$$L(x, \lambda, \mu) = F(x) + \lambda^T \varphi(x) + \mu^T \psi(x)$$

$$\varphi(x) = 0_L \equiv \varphi(x) \le 0_L \cap -\varphi(x) \le 0_L$$

$$\mathscr{D}_x = \{ x \in \mathscr{R}^S : \varphi(x) \le 0_L \cap -\varphi(x) \le 0_L, \psi(x) \le 0_M \}$$

$$L(x, \lambda', \lambda, \mu) = F(x) + \lambda^T \varphi(x) - \lambda'^T \varphi(x) + \mu^T \psi(x)$$

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Analytical methods

Disadvantages

It is hard to apply them if:

F, φ, ψ are nonlinear

$\dim(x)$ is large

They cannot be applied if:

F, φ, ψ are not differentiable

F is not given by formula and it may only be measured for requested value of x

Thank you for attention