Computer Science

Jerzy Świątek

Systems Modelling and Analysis

Choose yourself and new technologies

L.7. Noised measurements of the physical values

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Wrocław University of Technology

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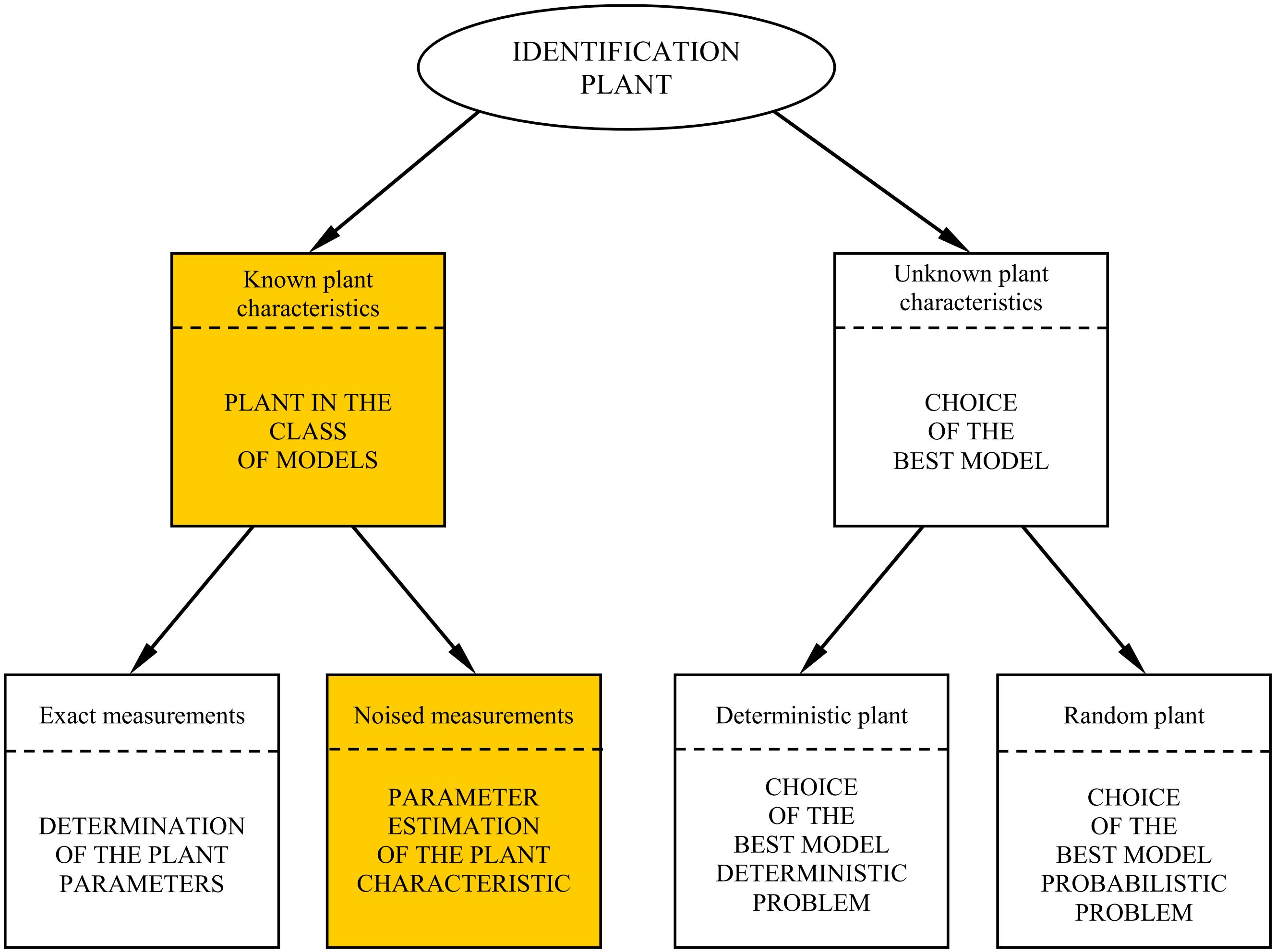
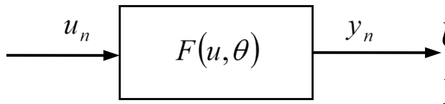
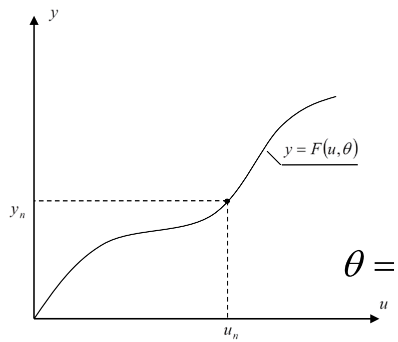

IDENTIFICATION
PLANT
Known plant
characteristics
PLANT IN THE
CLASS
OF MODELS
Unknown plant
characteristics
CHOICE
OF THE
BEST MODEL
Exact measurements
DETERMINATION
OF THE PLANT
PARAMETERS
Noised measurements
PARAMETER
ESTIMATION
OF THE PLANT
CHARACTERISTIC
Deterministic plant
CHOICE
OF THE
BEST MODEL
DETERMINISTIC
PROBLEM
Random plant
CHOICE
OF THE
BEST MODEL
PROBABILISTIC
PROBLEM

Plant in the class of model

Deterministyczny

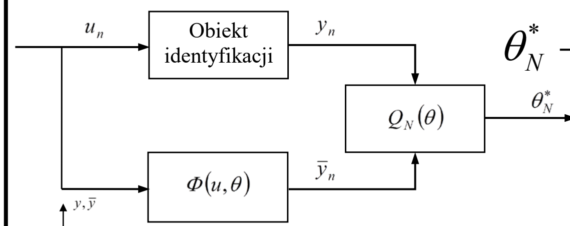


$$U_N=[u_1 \quad u_2 \quad \cdots \quad u_N],$$
$$Y_N=[y_1 \quad y_2 \quad \cdots \quad y_N]$$

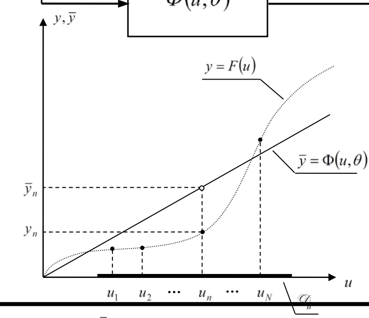


$$y_n=F(u_n,\theta),$$
$$n=1,2,\ldots,N$$
$$Y_N=\overline{F}(U_N,\theta)$$
$$\theta=\overline{F}^{-1}(U_N,Y_N)\overset{df}{=}\Psi_N(U_N,Y_N)$$

Choice of the best model



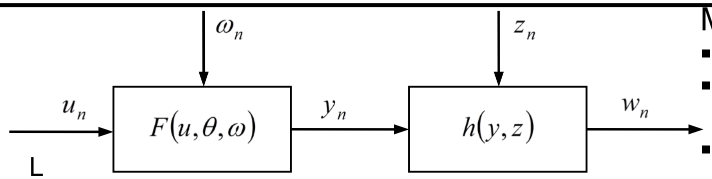
$$\bar{y}=\Phi(u,\theta_N^*)$$
$$\theta_N^* \rightarrow Q_N(\theta_N^*)=\min_{\theta} Q_N(\theta)$$
$$\theta_N^*=\Psi_N(U_N,Y_N)$$
$$U_N=[u_1 \quad u_2 \quad \cdots \quad u_N],$$
$$Y_N=[y_1 \quad y_2 \quad \cdots \quad y_N]$$



Optimal model for:

- Given measurements sequences
- Klass of models
- Given performance index

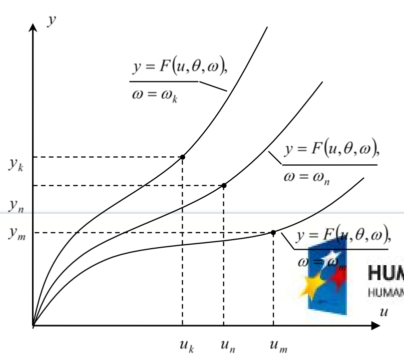
Losowy

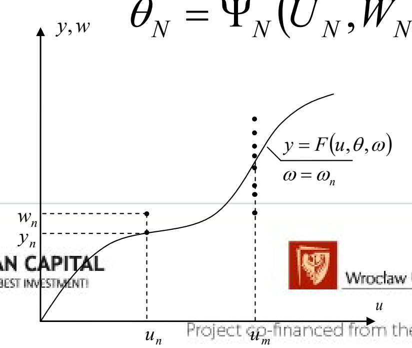


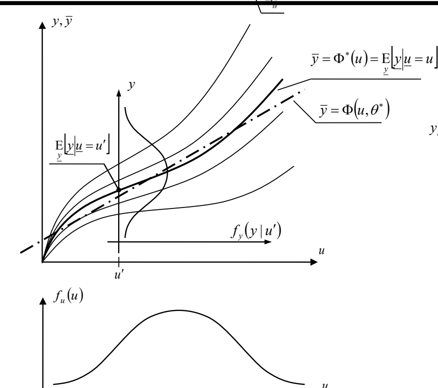
Methods:

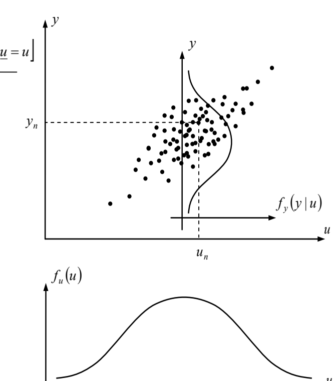
- Least square
- Maximum likelihood
- Bayes'a

$$\theta_N=\Psi_N(U_N,W_N)$$









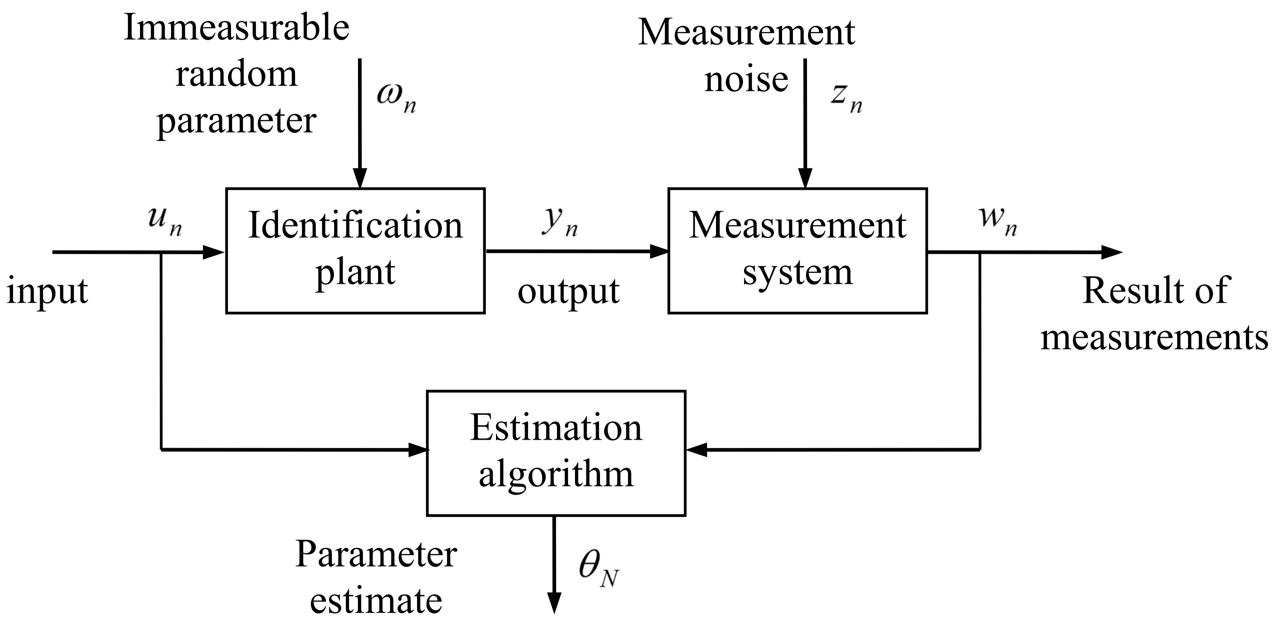
Full probabilistic knowledge

- First type regrsesion
- Second type regression

Unknown probabilostic knowledge

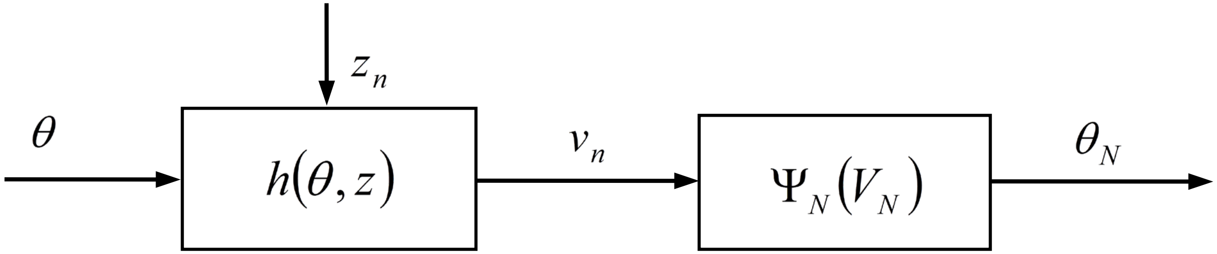
- Performance index estimatin
- Parameter estimation of the probability distribution
- Probabilit ditribution estimatoin

Plant parameter estimation problem



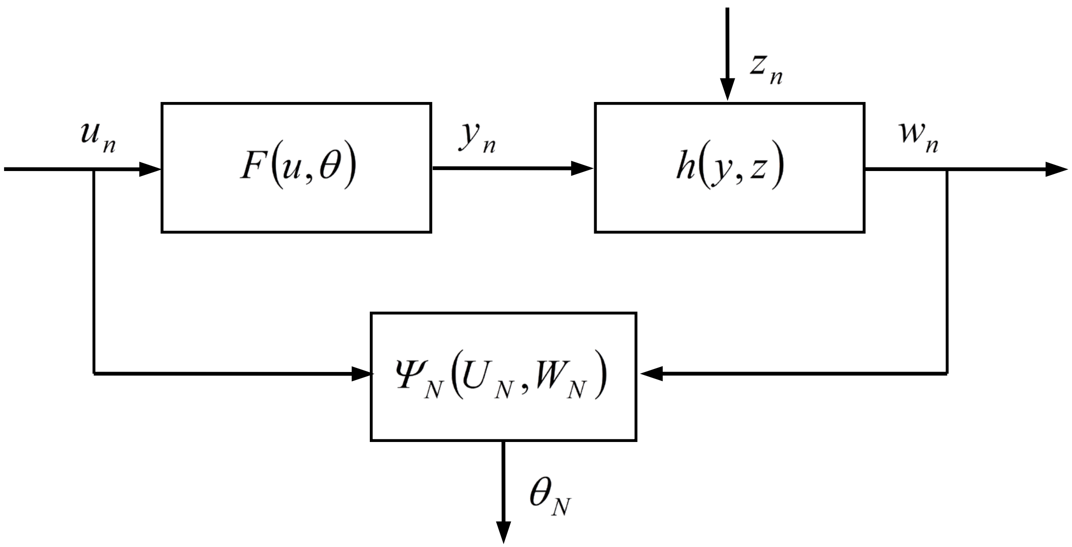
Plant parameter estimation problem

- Noised measurements of the physical values



Plant parameter estimation problem

- Deterministic plant, noised measurements of the plant output



where:

$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N]$$

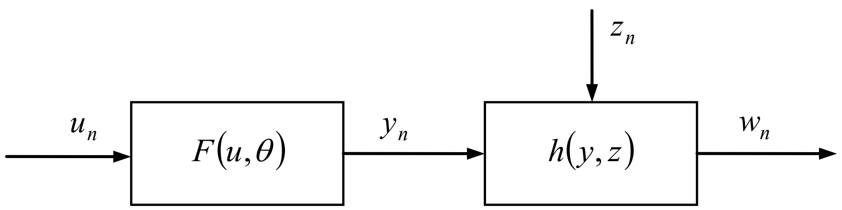
$$W_N = [w_1 \quad w_2 \quad \cdots \quad w_N]$$

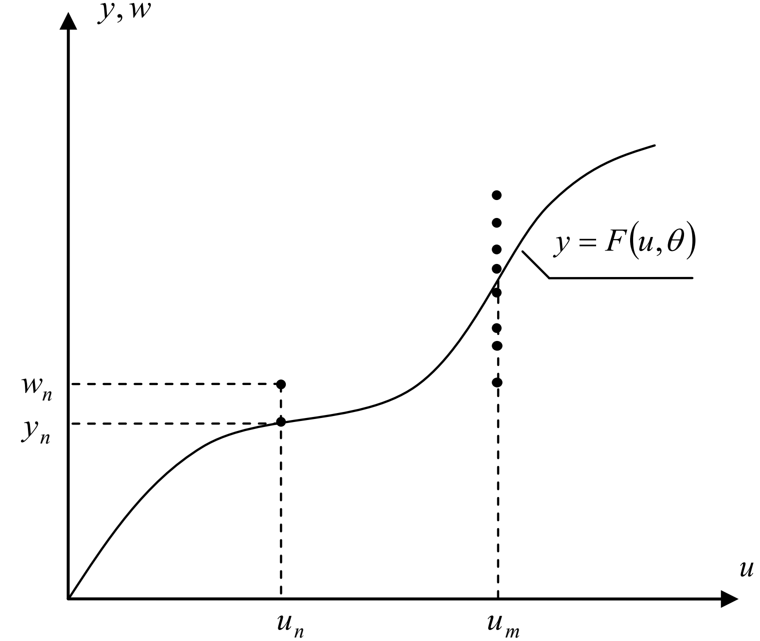
Ψ_N – estimation algorithm

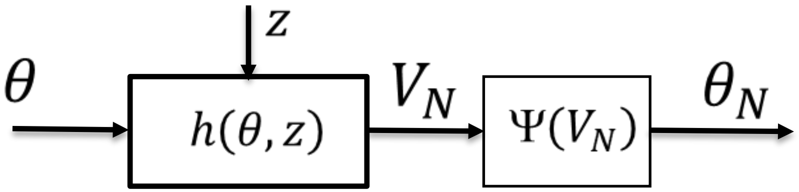
θ_N – estimate of θ

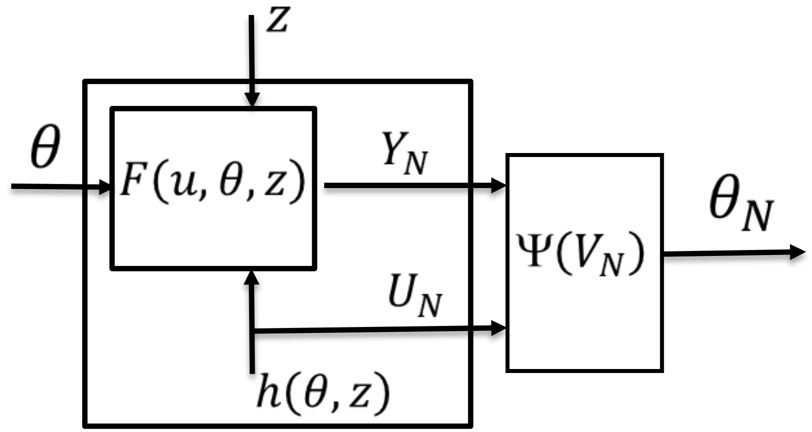
Deterministic plant, noised measurements of the plant output

- Noised measurements of the identification plant known static characteristics



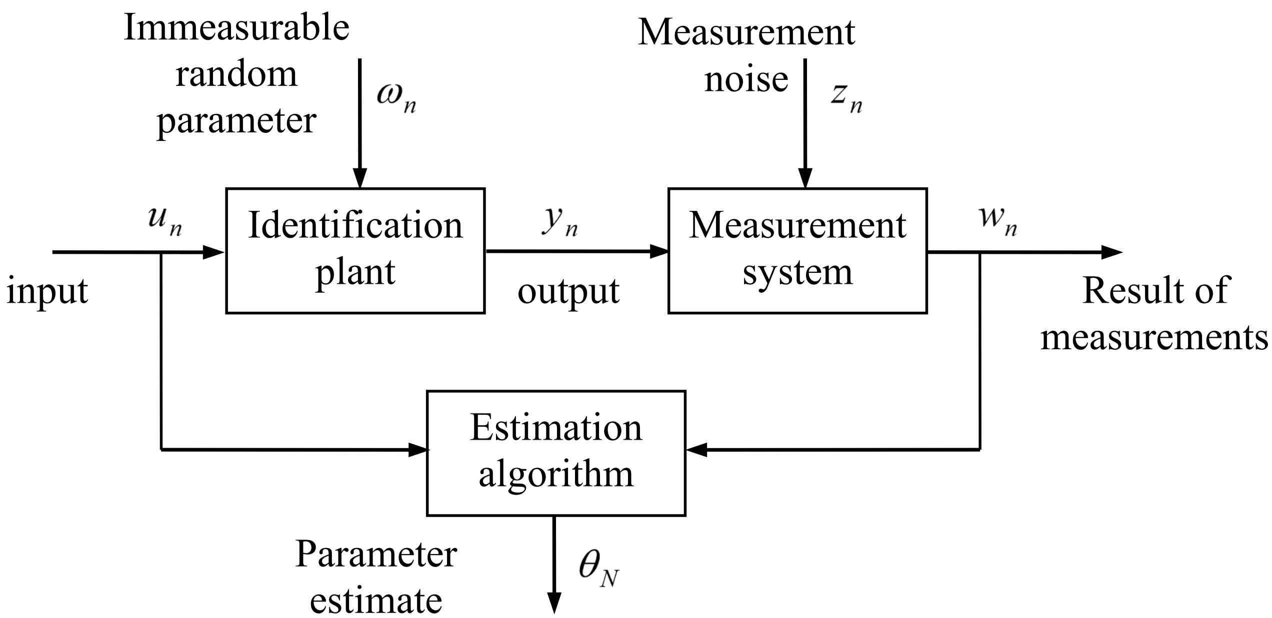






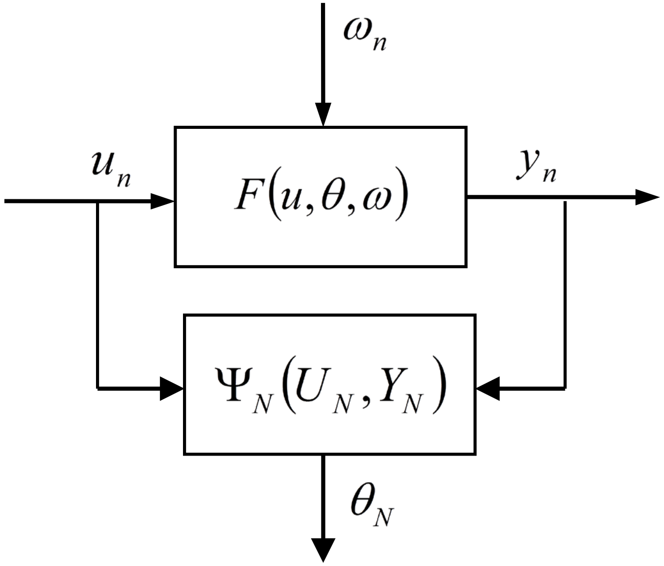
$$V_N = \begin{bmatrix} Y_N \\ U_N \end{bmatrix}$$

Plant parameter estimation problem



Plant parameter estimation problem

- Immeasurable random plant parameter



where:

$$U_N = [u_1 \quad u_2 \quad \cdots \quad u_N]$$

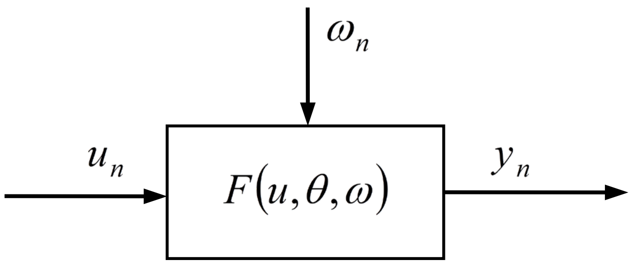
$$Y_N = [y_1 \quad y_2 \quad \cdots \quad y_N]$$

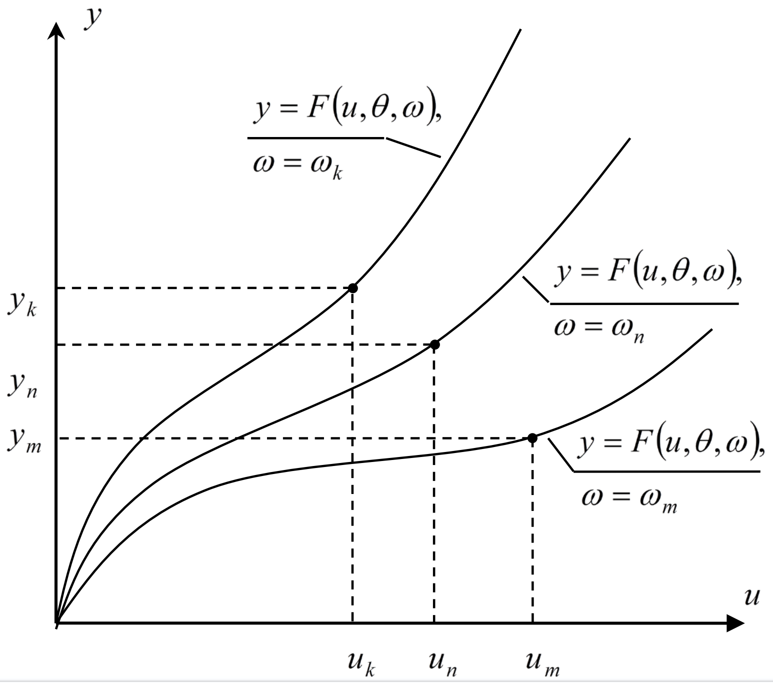
Ψ_N – estimation algorithm

θ_N – estimate of θ

Immeasurable random plant parameter

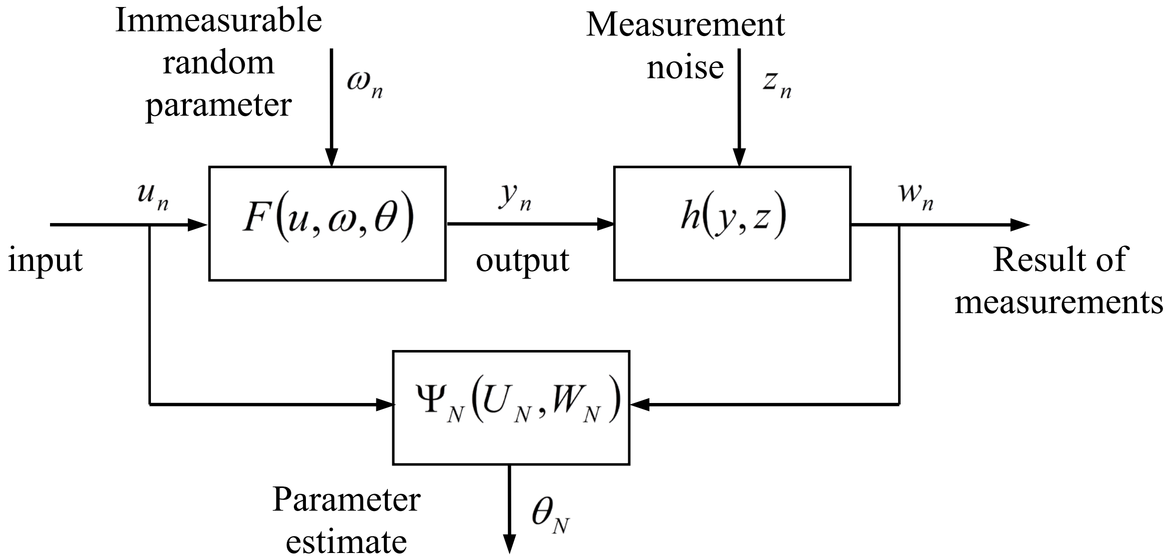
- Measurements of plant characteristic with random parameter



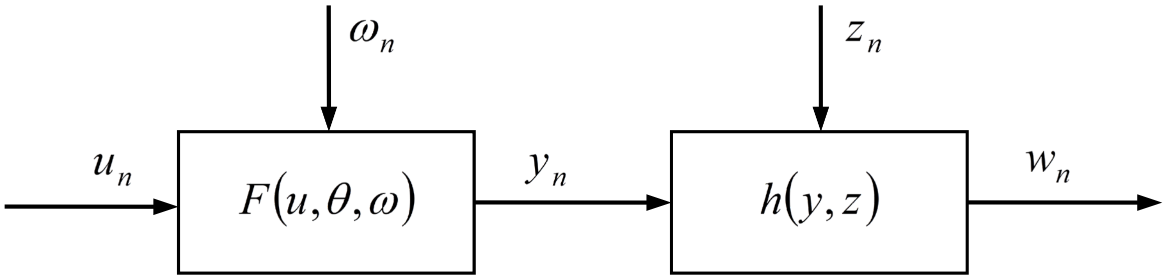


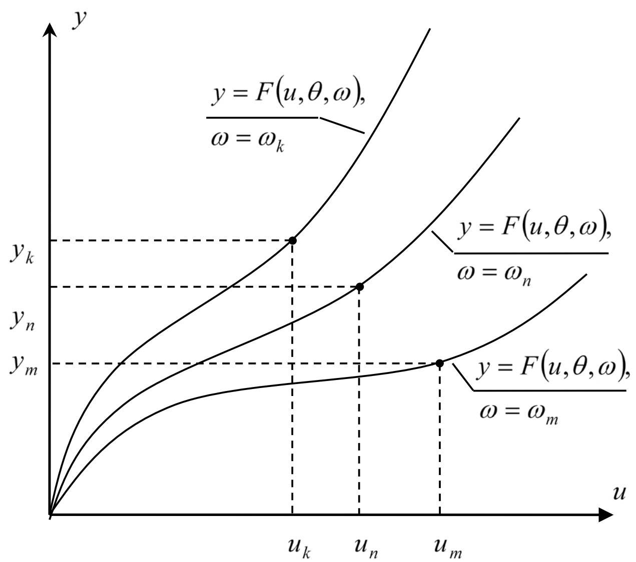
Plant parameter estimation problem

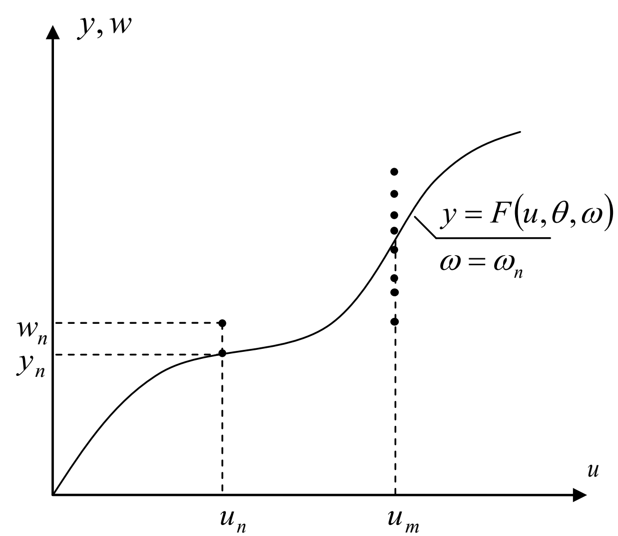
- Immeasurable random plant parameter and noised measurements of the plant output

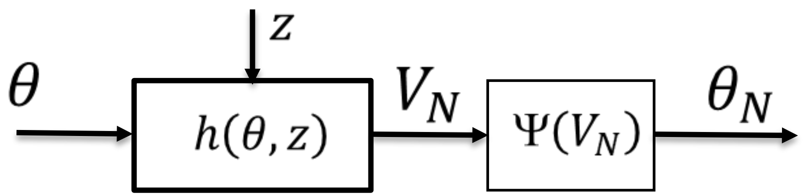


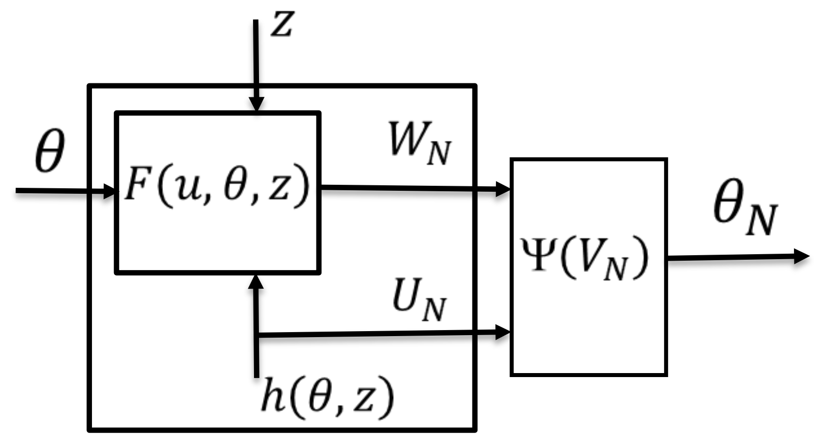
- Noised measurement plant output with randomly changed parameters







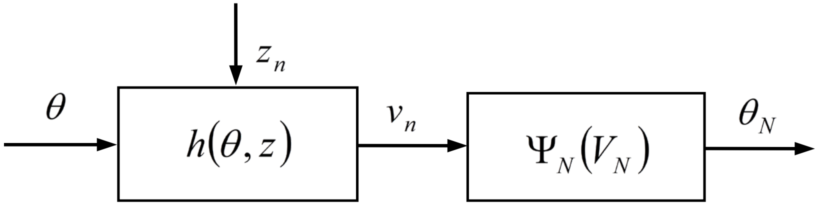




$$V_N \begin{bmatrix} W_N \\ U_N \end{bmatrix}$$

Noised measurements of the physical values

- Problem formulation



<u>Measurement system description:</u> $v = h(\theta, z)$

where: $v \in \mathscr{V}$, h – known one-to-one function

$h : \Theta \times \mathscr{Z} \to \mathscr{V}$, $z = h_z^{-1}(\theta, v)$

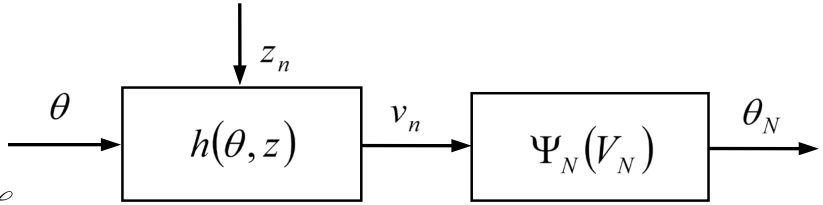
examples of h : $v = h(\theta, z) = \theta + z$

$v = h(\theta, z) = \theta \cdot z$

$\mathscr{W}$ – measurements domain ($\dim \theta = \dim z = L$)

Noised measurements of the physical values

- Problem formulation



<u>Measurement noise:</u>

z_n – value of random variable $\underline{z}$ from the space $\mathscr{Z}$

$f_z(z)$ – probability density function

θ – observed vector of parameters, value of random variable $\underline{\theta}$, $\theta \in \Theta \subseteq \mathscr{R}^R$

$f_\theta(\theta)$ – probability density function

Measurements: $V_N = [v_1 \quad v_2 \quad \cdots \quad v_N]$

Noised measurements of the physical values

General form of estimation algorithm:

$$\theta_N = \Psi_N\left(V_N\right)$$

- Solution:
  - Least square method
  - Maximum likelihood method
  - Bayesian method

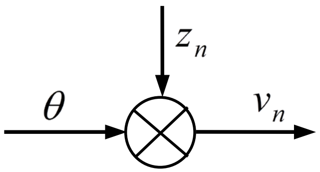
Least square method

<u>Assumptions:</u>

$v = h(\theta, z) = \theta + z$ – additive noise

$\underset{\underline{z}}{E}\left[\underline{z}\right] = 0$ – expected value of the noise signal is zero

$\underset{\underline{z}}{Var}\left[\underline{z}\right] < \infty$ – variance of the noise is not infinite



<u>Calculations:</u>

Least square method minimizes variance of noise signal:

$$Var_{zN}(V_N, \theta) = \frac{1}{N}\sum_{n=1}^{N}(v_n - \theta)^2$$

Estimation algorithm has the form:

$$\theta_N = \Psi_N(V_N) \quad \rightarrow \quad Var_{zN}(V_N, \theta_N) = \min_{\theta \in \Theta} Var_{zN}(V_N, \theta)$$

Estimation algorithm:

$$\theta_N = \frac{1}{N}\sum_{n=1}^{N} v_n$$

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Maximum likelihood method

x - Random variable

$f_x(x,\theta)$ - Probability density function

θ - Unknown parameter of density function

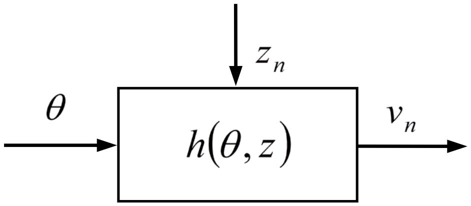
$[x_1, x_2, \cdots, x_N] \triangleq X_N$ - Sequence of random variables values

$L(X_N,\theta) \triangleq \prod_{n=1}^{N} f_x(x_n,\theta)$ - Likelihood function

θ_N - Estimate of unknown parameter θ

$$\theta_N = \Psi(X_N) \rightarrow L(X_N,\theta_N) = \max_{\theta} L(X_N,\theta)$$

Maximum likelihood method



Assumptions:

$\underline{v} = h(\theta, \underline{z})$ – measurement system is described by any one-to-one invertible function

Mathematical formula describing probability density function $f_z(z)$ is given.

Calculations:

Probability density function of observed value $\underline{v}$ with unknown parameter:

$f_v(v,\theta) = f_z\left(h^{-1}(\theta, v)\right) \cdot |J_h|,$ where J_h is Jacobi matrix of the inverse transformation.

Likelihood function has the form:

$$L_N(V_N, \theta) = \prod_{n=1}^{N} f_v(v_n, \theta) = \prod_{n=1}^{N} f_z\left(h_z^{-1}(\theta, v_n)\right)|J_h|, \quad \text{where:} \quad J_h = \frac{\partial h_z^{-1}(\theta, v)}{\partial v}$$

Maximum likelihood method

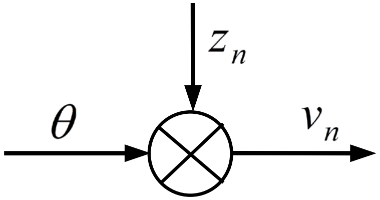
Estimation algorithm has the form:

$$\theta_N = \Psi_N(V_N) \quad \rightarrow \quad L_N(V_N, \theta_N) = \max_{\theta \in \Theta} L_N(V_N, \theta)$$

Maximum likelihood method

- Przykład 1

Measurement system:



$$v = h(\theta, z) = \theta + z$$

Probability density function :

$$f_z(z) = \frac{1}{\sigma_z \sqrt{2\pi}} \exp\left[-\frac{(z - m_z)^2}{2\sigma_z^2} \right]$$

Measurement system description:

$$v = h(a, z) = a + z \qquad z = h_z^{-1}(\theta, v) = v - \theta$$

Jacobi matrix:

$$J_h = \frac{\partial h_z^{-1}(\theta, v)}{\partial v} = \frac{d}{dv}(v - \theta) = 1$$

Maximum likelihood method

- Example 1

Probability density function:

$$f_v(v,\theta)=\frac{1}{\sigma_z\sqrt{2\pi}}\exp\left[-\frac{(v-\theta-m_z)^2}{2\sigma_z^2}\right]\cdot|1|$$

Likelihood function:

$$L_N(V_N,\theta)=\prod_{n=1}^{N}\frac{1}{\sigma_z\sqrt{2\pi}}\exp\left[-\frac{(v_n-\theta-m_z)^2}{2\sigma_z^2}\right]$$

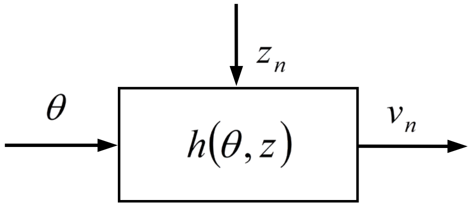
$$L_N(V_N,\theta)=\left(\frac{1}{\sigma_z\sqrt{2\pi}}\right)^N\exp\left[\sum_{n=1}^{N}-\frac{(v_n-\theta-m_z)^2}{2\sigma_z^2}\right]$$

Estimation algorithm:

$$\theta_N=\Psi_N(V_N)=\frac{1}{N}\sum_{n=1}^{N}(v_n-m_z)$$

Maximum likelihood method

- Example



Noise description: $$f_z(z) = \begin{cases} 1 & \text{for} \quad z \in [0,1] \\ 0 & \text{for} \quad z \notin [0,1] \end{cases}$$

Measurement system description: $$v = h(\theta, z) = \theta\, z \qquad (\theta > 0)$$

$$z = h_z^{-1}(\theta, v) = \frac{v}{\theta}$$

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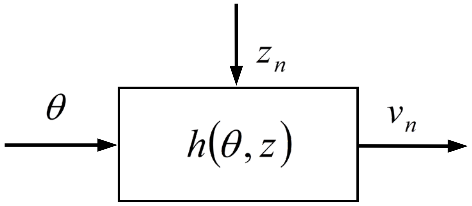
Maximum likelihood method

- Example



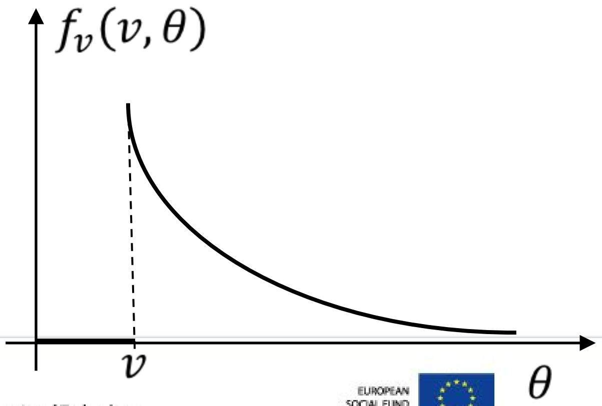
Jacobi matrix:

$$J_h = \frac{\partial h_z^{-1}(\theta, v)}{\partial v} = \frac{d}{dv}\left(\frac{v}{\theta}\right) = \frac{1}{\theta}$$

$$f_v(v,\theta) = \begin{cases} \frac{1}{\theta} & \text{for } \frac{v}{\theta} \in [0,1] \\ 0 & \text{for } \frac{v}{\theta} \notin [0,1] \end{cases}$$

Probability density function of the observed value:

$$f_v(v,\theta) = \begin{cases} \frac{1}{\theta} & for\ v \geq \theta \\ 0 & for\ v < \theta \end{cases}$$



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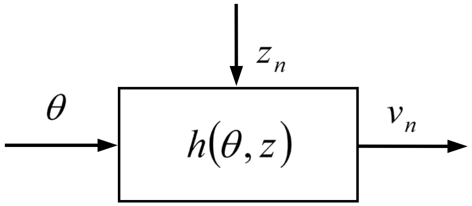
Maximum likelihood method

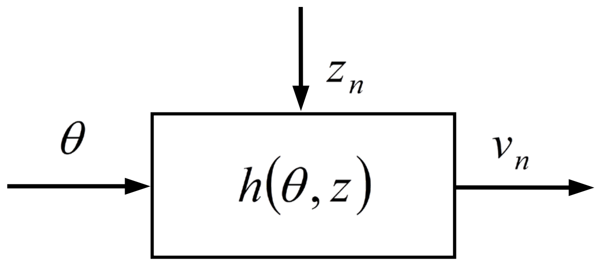
- Example



Likelihood function :

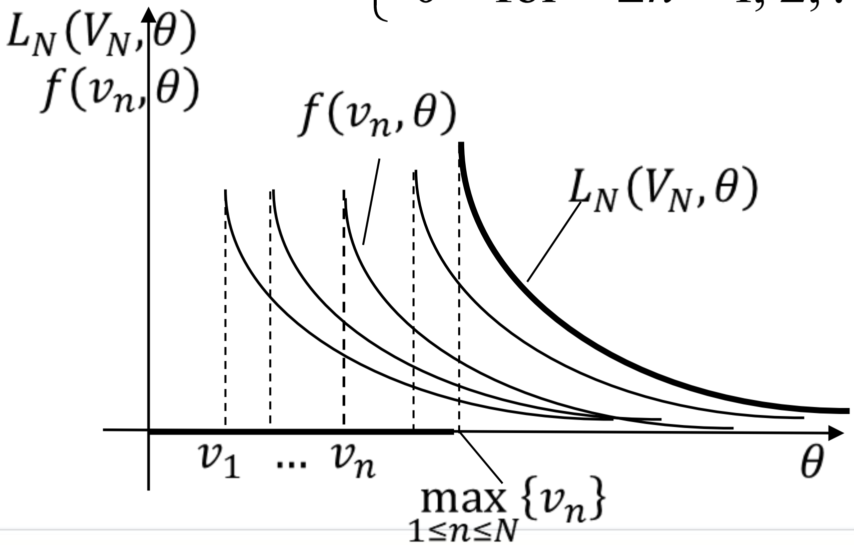
$$L_N(V_N, \theta) = \begin{cases} \dfrac{1}{\theta^N} & \text{for} \quad \forall n = 1, 2, \ldots, N \quad v_n \in [0, \theta] \\ 0 & \text{for} \quad \exists n = 1, 2, \ldots, N \quad v_n \notin [0, \theta] \end{cases}$$

$$L_N(V_N, \theta) = \begin{cases} \dfrac{1}{\theta^N} & for \quad \theta \geq \max\limits_{1 \leq n \leq N} \{v_n\} \\ 0 & for \quad \theta < \max\limits_{1 \leq n \leq N} \{v_n\} \end{cases}$$



Likelihood function:

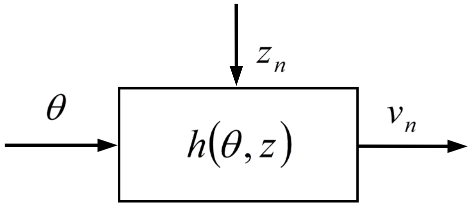
$$L_N(V_N, \theta) = \begin{cases} \dfrac{1}{\theta^N} & \text{for} \quad \forall n = 1, 2, \ldots, N \quad v_n \in [0, \theta] \\ 0 & \text{for} \quad \exists n = 1, 2, \ldots, N \quad v_n \notin [0, \theta] \end{cases}$$



$$L_N(V_N, \theta) = \begin{cases} \dfrac{1}{\theta^N} & \textit{for} \quad \theta \ge \max_{1 \le n \le N} \{v_n\} \\ 0 & \textit{for} \quad \theta < \max_{1 \le n \le N} \{v_n\} \end{cases}$$

Maximum likelihood method

- Example



Estimation algorithm:

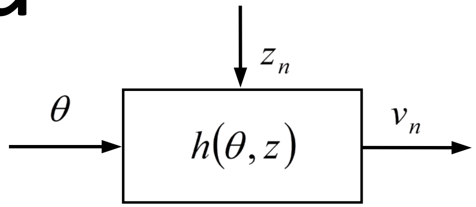
$$\theta_N = \Psi_N(V_N) = \max_{1 \le n \le N} \{v_n\}$$

Interpretation:

$$\theta_N = \max_{1 \le n \le N} \{v_n\} = \max_{1 \le n \le N} \{\theta\, z_n\} = \theta \max_{1 \le n \le N} \{z_n\}$$

Bayesian method

$$\bar{\theta}_N = \Psi(V_N)$$



Assumptions:

$\underline{v} = h(\theta, \underline{z})$ – measurement system is described by any one-to-one invertible function

Mathematical formulas describing probability density functions $f_z(z)$ and $f_\theta(\theta)$ are given.

The loss function $L(\theta, \bar{\theta})$ is defined, where $\bar{\theta}$ is estimed value of unknown parameter.

Calculations:

Risk: $$R(\overline{\Psi}) \overset{df}{=} \underset{\underline{\theta}, \underline{V_N}}{E}\left[L\left(\underline{\theta}, \underline{\bar{\theta}} = \overline{\Psi}(\underline{V_N})\right)\right] = \int_{\mathscr{V}_N}\int_{\Theta} L(\theta, \overline{\Psi}(V_N)) f(\theta, V_N)\, d\theta\, dV_N$$

where $f(\theta, V_N)$ is joint probability density function:

$$f(\theta, V_N) = f'(\theta | V_N) f''(V_N)$$

where f' is conditional probability density function and f'' is marginal probability density function

Bayes approach

$\theta \; - \; value\ of\ random\ variable\ \boldsymbol{\theta}$

$\theta \; - continuous\ random\ variable - \; \theta \in \Xi \subseteq \mathscr{R}^{R}$

$f_{\theta}(\theta)$ – probability density function

$\theta \; - discrete\ type\ random\ variable - \; \theta \in \Xi = \{\theta_1, \theta_2, \cdots, \theta_K,\}$

$p_k = P(\boldsymbol{\theta} = \theta_k), k = 1, 2, \cdots, K$ – probability density function

$\bar{\theta}$ - *possible decision*

$L(\theta, \bar{\theta}) - Loss\ function, i.e.:$

$$L(\theta, \bar{\theta}) \triangleq (\theta - \bar{\theta})^2$$

$$L(\theta, \bar{\theta}) \triangleq |\theta - \bar{\theta}|$$

$$L(\theta, \bar{\theta}) \triangleq -\delta(\theta - \bar{\theta}), -\delta_n(\theta - \bar{\theta})$$

Bayes approach

$R(\bar{\theta})=E_{\boldsymbol{\theta}}\left(L(\boldsymbol{\theta},\bar{\theta})\right)$ - Risk

$R(\bar{\theta})=\int_{\Xi} L(\theta,\bar{\theta})f_{\theta}(\theta)d\theta$ For continuous random variable

$R(\bar{\theta})=\sum_{k=1}^{K} L(\theta_k,\bar{\theta})p_k$ For discrete type random variable

$\theta^* \rightarrow R(\theta^*) = \min_{\bar{\theta}} R(\bar{\theta})$ Optimal decision

For: $L(\theta,\bar{\theta}) \triangleq (\theta-\bar{\theta})^2$

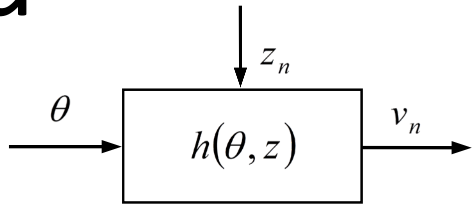
$$R(\bar{\theta})=\int_{\Xi} (\theta-\bar{\theta})^2 f_{\theta}(\theta)d\theta \rightarrow \theta^* = \int_{\Xi} \theta f_{\theta}(\theta)d\theta = E_{\boldsymbol{\theta}}(\boldsymbol{\theta})$$

For: $L(\theta,\bar{\theta}) \triangleq -\delta(\theta-\bar{\theta})$

$$R(\bar{\theta})=\int_{\Xi} -\delta(\theta-\bar{\theta})f_{\theta}(\theta)d\theta = -f_{\theta}(\bar{\theta}) \rightarrow \theta^* \rightarrow \min_{\bar{\theta}}\left(-f_{\theta}(\bar{\theta})\right) = \max_{\bar{\theta}} f_{\theta}(\bar{\theta})$$

Bayesian method

$$\bar{\theta}_N = \Psi(V_N)$$



<u>Assumptions:</u>

$\underline{v} = h(\theta, \underline{z})$ – measurement system is described by any one-to-one invertible function

Mathematical formulas describing probability density functions $f_z(z)$ and $f_\theta(\theta)$ are given.

The loss function $L(\theta, \bar{\theta})$ is defined, where $\bar{\theta}$ is estimed value of unknown parameter.

<u>Calculations:</u>

Risk: $$R(\overline{\Psi}) \overset{df}{=} \underset{\underline{\theta}, \underline{V_N}}{E}\left[L\left(\underline{\theta}, \underline{\bar{\theta}} = \overline{\Psi}(\underline{V_N})\right)\right] = \int\limits_{\mathcal{V}_N}\int\limits_{\Theta} L(\theta, \overline{\Psi}(V_N)) f(\theta, V_N)\, d\theta\, dV_N$$

where $f(\theta, V_N)$ is joint probability density function:

$$f(\theta, V_N) = f'(\theta | V_N) f''(V_N)$$

where f' is conditional probability density function and f'' is marginal probability density function

Bayesian method

The problem: $\Psi_N \quad \rightarrow \quad R(\Psi_N) = \min_{\overline{\Psi}} R(\overline{\Psi})$

$$R(\overline{\Psi}) = \int_{\mathscr{V}_N} \int_{\Theta} L(\theta, \overline{\Psi}(V_N)) f'(\theta | V_N)\, d\theta\; f''(V_N)\, dV_N$$

$$r(\overline{\theta}, V_N) \stackrel{\text{df}}{=} \underset{\underline{\theta}}{E}\left[L(\underline{\theta}, \overline{\theta}) | V_N\right] = \int_{\Theta} L(\theta, \overline{\theta} = \overline{\Psi}(V_N)) f'(\theta | V_N)\, d\theta$$

r – conditional risk

where:

Bayesian method

The problem is reduced to the equivalent one:

$$\theta_N = \Psi_N(V_N) \quad \rightarrow \quad r(\theta_N, V_N) = \min_{\bar{\theta} \in \Theta} r(\bar{\theta}, V_N)$$

$$f'(\theta|V_N) = \frac{f_\theta(\theta)\prod_{n=1}^{N} f_z\left(h_z^{-1}(\theta, v_n)\right)|J_h|}{\int_\Theta f_\theta(\theta)\prod_{n=1}^{N} f_z\left(h_z^{-1}(\theta, v_n)\right)|J_h|\, d\theta} = \frac{f_\theta(\theta)\prod_{n=1}^{N} f_z\left(h_z^{-1}(\theta, v_n)\right)|J_h|}{const}$$

$$\int_\Theta f_\theta(\theta)\prod_{n=1}^{N} f_z\left(h_z^{-1}(\theta, v_n)\right)|J_h|\, d\theta = const \;\; for \; given \; sequence \; V_N$$

$$f'(\theta|V_N) \;\; \alpha \;\; f_\theta(\theta)\prod_{n=1}^{N} f_z\left(h_z^{-1}(\theta, v_n)\right)|J_h|$$

where:

Bayesian method

- Example

Noise description: $$f_z(z) = \frac{1}{\sigma_z\sqrt{2\pi}}\exp\left[-\frac{z^2}{2\sigma_z^2}\right]$$

A priori distribution: $$f_\theta(\theta) = \frac{1}{\sigma_\theta\sqrt{2\pi}}\exp\left[-\frac{(\theta - m_\theta)^2}{2\sigma_\theta^2}\right]$$

Measurement system description: $$v = h(\theta, z) = \theta + z, \quad z = h_z^{-1}(\theta, v) = v - \theta$$

Loss function: $$L(\theta, \bar{\theta}) = -\delta(\theta - \bar{\theta})$$

Bayesian method

- Example

Jacobi matrix: $$J_h = \frac{\partial h_z^{-1}(\theta, v)}{\partial v} = \frac{d}{dv}(v-\theta) = 1$$

Probability density function of the observed value: $$f_v(v|\theta) = \frac{1}{\sigma_z\sqrt{2\pi}}\exp\left[-\frac{(v-\theta)^2}{2\sigma_z^2}\right]\cdot|1|$$

A posteriori probability density function:

$$f^{'}(\overline{\theta}|V_N) \quad \alpha \quad f_\theta(\overline{\theta})\prod_{n=1}^{N} f_z\left(h_z^{-1}(\theta, v_n)\right)|J_h| =$$

$$= \frac{1}{\sigma_\theta\sqrt{2\pi}}\exp\left[-\frac{(\overline{\theta}-m_\theta)^2}{2\sigma_\theta^2}\right]\prod_{n=1}^{N}\frac{1}{\sigma_z\sqrt{2\pi}}\exp\left[-\frac{(v_n-\overline{\theta})^2}{2\sigma_z^2}\right]$$

Bayesian method

- Example

For loss function: $L(\theta,\overline{\theta})=-\delta(\theta-\overline{\theta})$ the conditional risk

$$r(\overline{\theta},V_N)\overset{\text{df}}{=}\underset{\underline{\theta}}{E}\left[L(\underline{\theta},\overline{\theta})\middle|V_N\right]=\int_{\Theta}L\left(\theta,\overline{\theta}=\overline{\Psi}(V_N)\right)f'(\theta|V_N)\,d\theta=-f'(\overline{\theta}|V_N)\quad\alpha$$

$$\alpha\quad -f_\theta(\overline{\theta})\prod_{n=1}^{N}f_z\left(h_z^{-1}(\overline{\theta},v_n)\right)|J_h|=-\frac{1}{\sigma_\theta\sqrt{2\pi}}\left(\frac{1}{\sigma_z\sqrt{2\pi}}\right)^N\exp\left[-\frac{(\overline{\theta}-m_\theta)^2}{2\sigma_\theta^2}-\sum_{n=1}^{N}\frac{(v_n-\overline{\theta})^2}{2\sigma_z^2}\right]$$

Bayesian method

- Example

Estimation algorithm:

$$\theta_N = \Psi_N(V_N) = \frac{m_\theta + \left(\frac{\sigma_\theta}{\sigma_z}\right)^2 \sum_{n=1}^{N} v_n}{1 + \left(\frac{\sigma_\theta}{\sigma_z}\right)^2 N}$$

Discussion:

1° N – small number

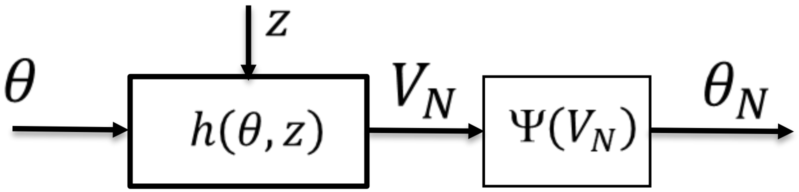
$(\sigma_z >> \sigma_\theta)$ – poor measurements

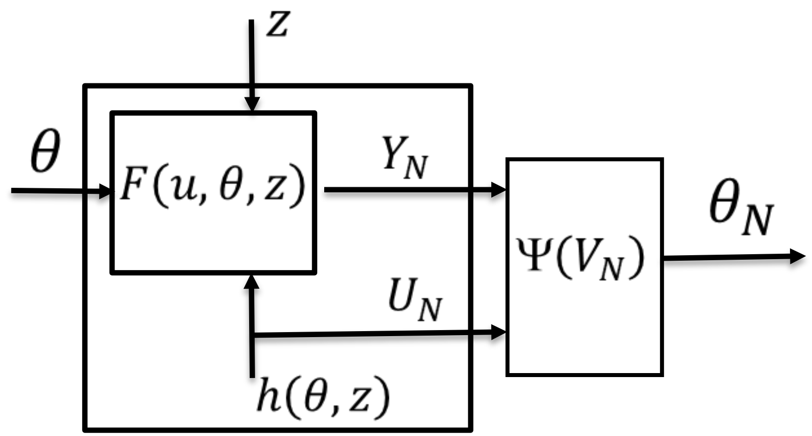
$$\theta_N \approx m_\theta$$

2° $N \to \infty$

$(\sigma_z << \sigma_\theta)$ – good measurements

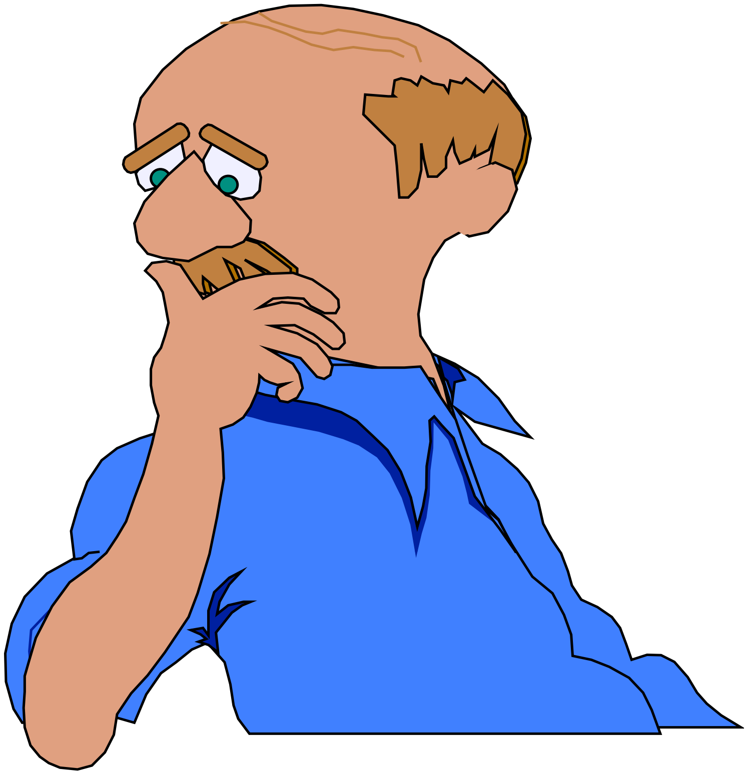
$$\theta_N \approx \frac{1}{N} \sum_{n=1}^{N} v_n$$





$$V_N = \begin{bmatrix} Y_N \\ U_N \end{bmatrix}$$

Thank you for attention



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